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THESIS

TIME SERIES MODELS WITH A SPECIFIED SYMMETRIC
NON-NORMAL MARGINAL DISTRIBUTION

by

Lee Samuel Dewald, Sr.

September 1985

Thesis Advisor: P.A.W. Lewis

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Time series models with autoregressive, moving average and mixed autoregressive-moving average correlation structure and with symmetric, heavy-tailed, non-normal marginal distributions, called ℓ-Laplace, are considered.

First, a flexible mixed model NLARMA(p,q) with Laplace (double exponential) marginals is investigated. The correlation structure for several special cases is derived. The innovation sequence for the second-order autoregressive case, NLAR(2), is derived. Parameter estimation in the NLAR(1) models is discussed in terms of moments, least squares and maximum likelihood.

Second, a family of continuous random coefficient models with ℓ-Laplace distributions are examined. The ℓ-Laplace distribution is described along with a useful transformation. The correlation structure for special cases is derived. For a special case when ℓ is one, the BELAR(1) model with Laplace marginals, the maximum likelihood estimator of serial correlation is derived. Least squares estimates are also derived using the concept of a linear residual. These estimators of correlation, along with other estimators of location and scale are compared in a small simulation study.

Thirdly, the NLAR(1) and the BELAR(1) processes are compared using higher-order residual analyses based on the uncorrelated, but dependent linear residuals, $\{R_n\}$.

Finally, open problems, as well as possible extensions and applications of the analyses given in this thesis are discussed.

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Time Series Models with a Specified Symmetric Non-Normal Marginal Distribution

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I. INTRODUCTION

In standard time series analysis, one assumes the marginal distributions of $\{X_n\}$ are Normal, i.e. Gaussian. However, a Gaussian distribution will not always be appropriate. In earlier works by Gaver and Lewis [Ref. 1]; Jacobs and Lewis [Refs. 2,3]; and Lawrance and Lewis [Refs. 4,5,6], stationary non-Gaussian time series models were developed for variables with positive and highly skewed marginal distributions.

There still remain other situations for which Gaussian marginals are inappropriate, i.e. the marginal time series variable being modelled, although not skewed or inherently positive valued, has a large kurtosis or long-tailed distribution. The position errors in a large navigation system have such a distribution. In particular, Hsu [Ref. 7] modelled pooled position errors using the double exponential distribution (also called the Laplace distribution). Also McGill [Ref. 8] showed that the Laplace distribution provides a characterization of the error in a timing device under periodic excitation. Speech-waves are modelled using Laplace variables (Davenport [Ref. 9]). In the "speech-like" process given by the linear AR(1) model

$$X_n = cX_{n-1} + (1 - c^2)^{1/2}E_n, \qquad \text{(I.1)}$$

where $.8 \leqq c \leqq .9$, the innovation sequence $\{E_n\}$ is i.i.d. Laplace (Linde and Gray [Ref. 10]). In image coding systems using a two-dimensional

discrete cosine, DC, transform, Reininger and Gibson [Ref. 11] showed that the Laplace distribution gives the best approximation to the distribution of the non-DC coefficients. Recently Sethia and Anderson [Ref. 12] required a stationary autoregressive process with Laplace marginals in their research in communications technology.

Even before Gaver and Lewis [Ref. 1] wrote the pioneering paper on the subject of autoregressive processes with a specified non-Normal marginal distribution, Gastwirth and Wolff [Ref.13] had derived a solution to the linear additive first-order difference equation

$$X_n = \rho X_{n-1} + E_n, \tag{I.2}$$

for which $\{X_n\}$ is marginally Laplace. This result was used later by Gastwirth and Rubin [Ref.14] within the context of robust estimation on dependent data. This solution to (I.2) is here called the Laplace First-order Autoregressive Process (LAR(1)). The early solution of (I.2) is mentioned at this point, merely to further substantiate the claim that non-Normal, heavy-tailed distributions are of interest.

In this thesis, several time series models with a specified symmetric, heavy-tailed marginal distribution are presented. This distribution, called the ℓ-Laplace distribution, includes the Laplace distribution as a special case. The approach in Chapter II extends the discrete random coefficient model of Lawrance and Lewis [Ref. 6], New Exponential Autoregressive Moving Average--NEARMA(p,q), to the case where the marginal distribution is Laplace, also called double

Exponential. This class of models is called The New Laplace Autoregressive Moving Average model, NLARMA(p,q). Several special cases of NLARMA(p,q) are individually researched. The second-order autoregressive model, NLAR(2), is established by showing the conditions for existence and uniqueness and by specifying the innovation structure. The correlation structure of NLAR(2) is also given along with results concerning directional moments and partial time reversibility.

For the case when p = 1 and q = 0, called NLAR(1), the distribution of the difference $X_n - X_{n-1}$ is derived, providing some insight into the nature of the differenced NLAR(1) model. The conditional density of X_n given X_{n-1} is also derived, which leads to a brief investigation of the likelihood function. Parameter estimation in NLAR(1), however, is limited to comparisons of the moment estimators and the least squares estimators for the independent model parameters of serial correlation.

The correlation structure is derived for other models in the NLARMA(p,q) family: the first-order moving average called NLMA(1); the first-order mixed model called NLARMA(1,1); and the special cases of p^{th}-order autoregressive models called TLAR(p) that are analogous to the TEAR(p) model of Lawrance and Lewis [Ref. 6]. These models demonstrate the flexibility of the NLARMA(p,q) family.

In Chapter III, a family of stationary time series is developed using continuous random coefficients in the additive difference equation model. The marginal distribution is specified to be a member of the so-called ℓ-Laplace distributions, the properties of which are described at

the beginning of the chapter. The "square-root Beta-Laplace" transform is defined. It is used to formulate the ℓ-Laplace time series models.

For the special case when $\ell = 1$, the marginal distribution is again Laplace. The autoregressive model is called the Beta-Laplace First-Order Autoregressive model, BELAR(1). The conditional density of X_n given X_{n-1} is derived. This leads to the derivation of a likelihood function and a numerical technique to evaluate and maximize the likelihood function with respect to the model parameter for serial correlation.

Several facets of the parameter estimation problem are investigated for BELAR(1). The behavior of different estimators of scale and location are compared using the Simulation Testbed (SIMTBED) of Lewis, Orav and Uribe [Ref. 15]. The least squares estimation theory is derived around the concept of a linearized residual. Asymptotic properties are derived using results from Nicholls and Quinn [Ref. 16]. Robust estimators are defined and simulated in SIMTBED. Finally, a numerical scheme for finding the maximum likelihood estimator of serial correlation is used in a small simulation study of the small sample properties of the maximum likelihood estimator.

In the last section of Chapter III, a first-order moving average model is discussed. A q^{th}-order moving average model in ℓ-Laplace variables is also derived.

The random coefficient approaches are not the only ways to generate Laplace or other variables with a specified correlation structure. The literature contains numerous articles on generation of random sequences.

One approach put forth in several papers (Gujar and Kavanagh [Ref. 17]; Haddad and Valisalo [Ref. 18]; Li and Hammond [Ref. 19]; Liu and Munson [Ref. 20]; Sondhi [Ref. 21]) involves passing white Gaussian noise through a linear filter followed by a zero memory nonlinear transform. This is a general procedure that produces exactly the required marginal distribution and a good approximation to the autocorrelation structure. However, the scheme lacks the simplicity of either of the methods being proposed. Moreover, the filtering approach produces, for example, in the first-order autoregressive case, only <u>one</u> process.

It is important to note that in non-Normal time series, there are infinitely many processes with a given marginal and autocorrelation structure. This is the case, for example, in the two-parameter NLAR(1) process. The differences in these processes must be explored through higher joint moments. In Chapter IV, residual analyses using fourth joint moments are derived. The ideas are modifications of those from Lawrance and Lewis [Refs. 6, 22], who accomplished an analysis using joint third moments within the NEAR framework. The residual analysis is applied to show the differences in the various NLAR(1) processes and the BELAR(1) process.

In Chapter V, open problems and possible extensions of the analyses given in this thesis are discussed. Possible applications to the analysis of wind velocity data are detailed.

II. DISCRETE RANDOM COEFFICIENT MODELS WITH LAPLACE MARGINALS

A. INTRODUCTION

Two aspects of modelling with dependent random variables are widely studied--the marginal distribution and the correlation structure. It is widely known how to generate sequences with either a specified marginal distribution or a particular correlation structure. Transforming the random variables may have an undesirable and unknown effect on the correlation structure. Likewise, the marginal distribution of a filtered process may be unknown.

It is the generation of random variables with both a specified marginal and a specified correlation structure that is discussed in this chapter. Specifically, we want sequences with a Laplace (double Exponential) marginal distribution and with ARMA(p,q) correlation structures as given by Box and Jenkins [Ref. 23] for the usual linear ARMA(p,q) models.

The following is an example of a process that has Laplace marginals. Let $\{X_n\}$ be a binary Markov chain with transition matrix P, so that $P[X_n=0|X_{n-1}=0] = \alpha_1$, $P[X_n=1|X_{n-1}=0] = 1-\alpha_1$, $P[X_n=1|X_{n-1}=1] = \alpha_2$, and $P[X_n=0|X_{n-1}=1] = 1-\alpha_2$. Let $L_n = (-1)^{X_n}E_n$, where $\{E_n\}$ is an i.i.d. Exponential sequence. If $\alpha_1=\alpha_2=\alpha$, $\{L_n\}$ has a Laplace marginal distribution. However, the correlation structure is not that of an AR(1) process. It is, in fact, easy to see that $\mathrm{Corr}(L_n,L_{n-k}) =$

$(1/2)(2\alpha-1)^{|k|}$, for $k=\pm1,\pm2,\ldots$, which is not a pure geometric function of k.

Two processes which produce an AR(1) correlation structure and a Laplace marginal distribution are the Laplace Discrete AR(1), LDAR(1), which is an adaption of the DAR(1) process of Jacobs and Lewis [Ref. 2], and the linear process of Gastwirth and Wolff [Ref. 13], called the LAR(1) process. The LDAR(1) model produces an $\{X_n\}$ sequence using the first-order autoregressive equation with random coefficients

$$X_n = V_n X_{n-1} + (1-V_n)L_n, \tag{II.A.1}$$

where $\{V_n\}$ is an i.i.d. sequence of Bernoulli random variables with $P\{V_n=1\} = 1-P\{V_n=0\} = \rho$; $\{L_n\}$ is an i.i.d. sequence of Laplace random variables. The coefficient and innovation processes from time n are assumed to be independent of $X_{n-1}, X_{n-2}, \ldots$. This sequence produces runs of constant value when successive realizations for V_n produces the value 1. When V_n is zero, a new value is selected. Although LDAR(1) is of limited value in general application because of this runs property, it is significant in that it is one of the first in a series of more general discrete random coefficient equation models for non-Normal time series, and it produces a first-order autoregressive Markovian process for <u>any</u> specified marginal distribution.

The LAR(1) model turns out to be a special case of the more general process called the New Laplace Autoregressive Moving Average model, NLARMA(p,q). Properties of the LAR(1) process are pointed out in the

next section of this chapter, which gives a characterization of the Laplace distribution.

The NLARMA(p,q) model is a very useful family of time series models that are discrete random coefficient linear difference equations. The models are extensions of the NEARMA(p,q) structure of Lawrance and Lewis [Refs. 4,5,6] to those cases where the underlying marginal distribution is Laplace rather than Exponential. The family provides great flexibility to systems modelling, because of the broad range of correlations and different dependency structures which are obtainable.

Section C is an examination of the second-order autoregressive model of the family, NLAR(2), for p = 2 and q = 0 in NLARMA(p,q). Conditions for the existence and uniqueness of the strictly stationary NLAR(2) model are derived using results from Nicholls and Quinn [Ref. 16] about Random Coefficient Autoregressive models of order k, RCA(k). In a proof, very similar to that given by Lawrance and Lewis for the NEAR(2) model [Ref. 6], the innovation for the NLAR(2) model is derived explicitly. The innovation is shown to be a convex combination of scaled Laplace variables. The correlation structure in the NLAR(2) model is shown to satisfy the Yule-Walker type equations just as do the linear AR(2) models. Aspects of directionality and time reversibility are also addressed.

In Section D, the first-order autoregressive model, NLAR(1), is described. It is a two-parameter, first-order Markov process which is a special case of the NLAR(2) model. The distribution of differences is derived. The conditional density of X_n given X_{n-1} and the likelihood

function are also derived. The non-differentiability of the likelihood function for all values of the two parameters has prevented the development of the maximum likelihood estimators. Parameter estimation is discussed within the context of moment estimators and least squares, using the usual linearized residual.

In Section E, several different special cases of NLARMA(p,q) are formulated and briefly discussed. The correlation structure for a first-order moving average model, NLMA(1), and a mixed autoregressive moving average model, NLARMA(1,1) are given. Correlation structure is derived and parameter estimation is discussed for the general p^{th}-order autoregressive models, TLAR(p), which are special cases of the NLAR(p).

Each of these models in Section E could well be the basis for further research. The intent at this point is primarily to further substantiate the claim of wide versatility and tractability in modelling non-Normal time series within the context of the NLARMA(p,q) family.

For example, the bivariate AR(1) process with Exponential marginal distributions of Dewald and Lewis [Ref. 24], can be extended to the case where the marginal distribution is Laplace. This, however, is not discussed further in this thesis.

B. CHARACTERIZATION OF THE LAPLACE DISTRIBUTION

1. Properties of the Laplace Distribution

The Laplace distribution is also known as the double Exponential distribution. In general, the density of a Laplace distributed variable, L, has two parameters--a location parameter $-\infty < \mu < +\infty$, and a

scale parameter $\lambda > 0$. The parameter μ is fixed here at zero. For $-\infty < x < \infty$ we have

$$f_L(x;\lambda) = \frac{1}{2\lambda} \exp(-|x|/\lambda). \qquad \text{(II.B.1.1)}$$

In what follows we will define $\{L_n\}$ as a sequence of i.i.d. random variables of the Laplace distribution with $\lambda = 1$ (Standard Laplace). The characteristic function of the standard Laplace variable is

$$\phi_L(\omega) = \frac{1}{1 + \omega^2}, \quad -\infty < \omega < \infty, \qquad \text{(II.B.1.2)}$$

and we have

$$E(L^n) = \begin{cases} 0 & \text{if } n \text{ is odd,} \\ n! & \text{if } n \text{ is even,} \end{cases} \qquad \text{(II.B.1.3)}$$

so that $E(L) = 0$, $Var(L) = 2$, skewness is zero, and kurtosis is 3. The value of the kurtosis indicates that the symmetric Laplace distribution has heavier tails than the normal distribution, for which the kurtosis is 0.

The sum of $n \geqq 2$ i.i.d. standard Laplace variables can be written as the difference of two i.i.d. random variables Y_1, Y_2 with Gamma distribution, shape parameter $k = n$ and scale parameter $\lambda = 1$.

This follows immediately from the characteristic function. Let $Y = \sum_{i=1}^{n} L_i$; then

$$\phi_Y(\omega) = \left\{\frac{1}{1 + \omega^2}\right\}^n = \left\{\frac{1}{1 + i\omega}\right\}^n \left\{\frac{1}{1 - i\omega}\right\}^n = \phi_{Y_1}(\omega)\, \phi_{Y_2}(-\omega). \qquad \text{(II.B.1.4)}$$

This result is quickly generalized. Replacing n by $t > 0$, we see that $[\phi_L(\omega)]^t$ is the characteristic function for the variable $X = Y_1 - Y_2$ where $Y_i \sim \text{Gamma}(t,1)$, $i = 1,2$ and Y_1 and Y_2 are independent. This demonstrates that the Laplace distribution is infinitely divisible.

Another useful result is obtained from (II.B.1.4) when $n = 1$. It shows that a Laplace variable is the difference of two i.i.d. exponential ($\lambda = 1$) variables. This makes it quite simple to generate Laplace distributed variates in computer simulations.

Random variables with a standard Laplace distribution are self-decomposable. Let

$$\phi_\varepsilon(\omega) = \phi_L(\omega)/\phi_L(\rho\omega), \qquad 0 \leqq \rho < 1. \qquad \text{(II.B.1.5)}$$

According to Feller [Ref. 25: p. 588], if $\phi_\varepsilon(\omega)$ is the transform of a random variable for each $0 \leqq \rho < 1$, then L is said to be self-decomposable. But for $-\infty < \omega < \infty$

$$\begin{aligned} \phi_\varepsilon(\omega) &= \{1 + (\rho\omega)^2\}(1 + \omega^2)^{-1}, \\ &= \{\rho + (1 - \rho)(1 - i\omega)^{-1}\}\{\rho + (1 - \rho)(1 + i\omega)^{-1}\} \qquad \text{(II.B.1.6)} \end{aligned}$$

$$= \rho^2 + (1 - \rho^2)(1 + \omega^2)^{-1}. \qquad \text{(II.B.1.7)}$$

We recognize (II.B.1.6) as the product of the characteristic functions of two i.i.d. innovation variables, ε_1 and $-\varepsilon_2$, as described in the EAR(1) process in [Ref. 1]. Also from (II.B.1.7)

$$\varepsilon = \begin{cases} 0 & \text{w.p.} \quad \rho^2, \\ L & \text{w.p.} \quad 1 - \rho^2. \end{cases} \qquad \text{(II.B.1.8)}$$

2. The Laplace First-Order Autoregressive Process, LAR(1)

The i.i.d. sequence $\{\varepsilon_n\}$ with distribution given in (II.B.1.8) is the innovation process of a first-order linear autoregressive equation

$$X_n = \rho X_{n-1} + \varepsilon_n, \qquad \text{(II.B.2.1)}$$

where $\{X_n\}$ is a stationary time series with double exponential marginal distribution, $|\rho|<1$. This is the LAR(1) model. It is actually a rediscovery in light of the fact that Gastwirth and Wolff [Ref. 13] had derived it earlier; also, Gastwirth and Rubin [Ref. 14] discuss it within the context of robust estimation techniques. The present account of LAR(1) includes new results.

The LAR(1) model has the same properties as the EAR(1) model in [Ref. 1] with two important differences. First, if $-1 < \rho < 0$, negative serial correlations for odd lags are obtained. Secondly, it is

partially time reversible in the sense that for all ℓ and n, both of the following are true:

$$E(X_n^2 X_{n+\ell}) = E(X_n X_{n+\ell}^2) = 0, \qquad \text{(II.B.2.2)}$$

$$P(X_n \geq X_{n-1}) = P(X_n \leq X_{n-1}) = 1/2. \qquad \text{(II.B.2.3)}$$

These results are derived in Section II.C and Section II.D. Note, however, that since LAR(1) is a linear AR(1) model with non-Gaussian innovation $\{\varepsilon_n\}$, it is not fully time reversible (Weiss [Ref. 26]). Also, note that this LAR(1) model has the zero defect property; when $\varepsilon_n = 0$, then $X_n/X_{n-1} = \rho$ and ρ can be determined exactly in long enough runs of the series $\{X_n\}$. This property is generally undesirable, but the broader NLAR(2) model developed in the next section is free of this defect, except for the special parameter values for which it reduces to the LAR(1) model.

If no repeats are observed in a realization of the time series, an extremely efficient estimator of ρ for LAR(1) is the median of the ratio X_i/X_{i-1}. The simulation results given in Table II.B.2.1 substantiate this claim. In Section II.D.4 and again in III.E.5, using the framework of the Simulation Testbed (SIMTBED) [Ref. 15], we will see that this median ratio is for small samples very biased, and is, apparently, asymptotically biased in all of the random coefficient AR(1) models with a Laplace marginal distribution that we examine.

TABLE II.B.2.1

Simulation Results using Median $\{X_i/X_{i-1}\}$ to Estimate

ρ in the LAR(1) Process for Samples of Size 2000

True ρ	$\hat{\rho} = \text{med}\ \{X_i/X_{i-1}\}$	Comments
-.9	-.9	-.9 occurred 1586 times in 1999 ratios
-.2	-.2	-.2 occurred 75 times in 1999 ratios
-.1	-.08746	-.1 occurred 11 times in 1999 ratios
+.01	+.01986	+.01 never occurred in 1999 ratios
+.5	+.5	+.5 occurred 490 times in 1999 ratios
+.75	+.75	+.75 occurred 1149 times in 1999 ratios

C. A SECOND ORDER AUTOREGRESSIVE LAPLACE TIME SERIES MODEL, NLAR(2)

1. Introduction

Using the terminology from [Ref. 6] the following time series model called NLAR(2), New Laplace Second-order Autoregressive model is proposed. This is a special case of NLARMA(p,q) model with p = 2, q = 0. The NLAR(2) model has four parameters, double exponential marginal distribution for $\{X_n\}$, second-order autoregressive Markov dependence, and autocorrelations satisfying Yule-Walker type equations.

The stationary NLAR(2) model has the same form as the stationary NEAR(2) model in [Ref. 6]. Writing the time series $\{X_n\}$ in the form of an additive, linear, random coefficient autoregressive difference equation, we have for all n that

$$X_n = \beta_1 K_n' X_{n-1} + \beta_2 K_n'' X_{n-2} + \varepsilon_n, \qquad \text{(II.C.1.1)}$$

where $\{K_n', K_n''\}$ is a sequence of i.i.d. discrete bivariate random variables with distribution

$$\{K_n', K_n''\} = \begin{cases} (1,0) & \text{w.p. } \alpha_1, \\ (0,1) & \text{w.p. } \alpha_2, \qquad n = 0, \pm 1, \pm 2, \ldots ; \\ (0,0) & \text{w.p. } 1-\alpha_1-\alpha_2, \end{cases} \qquad \text{(II.C.1.2)}$$

$\{\varepsilon_n\}$ is an i.i.d. innovation sequence whose distribution is given in (II.C.2.4); and $\{\varepsilon_n\}$ and $\{K_n', K_n''\}$ are mutually independent and independent of $X_{n-1}, X_{n-2}, \ldots$. The parameter space is defined by $0 \leqq |\beta_i| \leqq 1$ and $0 \leqq \alpha_i \leqq 1$, $i = 1,2$; $\alpha_1 + \alpha_2 \leqq 1$. Graphs of the admissible regions in the parameter space and the correlation space are presented in Section II.C.3.

Equations (II.C.1.1) and (II.C.1.2) have a direct physical interpretation. The observed process at time n, X_n, is only one of three possibilities: i) X_n is some multiple of what it was at time n-1, $\beta_1 X_{n-1}$, plus some random noise ε_n; ii) X_n is some multiple (possibly different than β_1), of its value at time n-2, $\beta_2 X_{n-2}$, plus some

independent random noise; iii) X_n is just random noise, ε_n, independent of everything up to time n.

2. Existence and Uniqueness

The work of Nicholls and Quinn [Ref. 16] on random coefficient autoregressive models is relevant to the NLAR(2) process. They have given the necessary and sufficient conditions for the existence of the unique covariance stationary solution to the following class of univariate random coefficient autoregressive models of order p, RCA(p):

$$Z_n = \sum_{i=1}^{p} \{\gamma_i + B_n(i)\} Z_{n-i} + \varepsilon_n, \qquad \text{(II.C.2.1)}$$

$n = 0, \pm 1, \pm 2, \ldots$, where

a. the γ_i's are real constants;

b. $\{\underset{\sim}{B}_n\}$ is a p-vector, second-order stationary, independent process with $E(\underset{\sim}{B}_n) = \underset{\sim}{0}$ and constant covariance matrix;

c. $\{\varepsilon_n\}$ is a scalar, second-order stationary, independent process, independent of $\{\underset{\sim}{B}_n\}$, with $E(\varepsilon_n^2) = \sigma^2$ for all n.

They also have shown that if $\{\underset{\sim}{B}_n\}$ and $\{\varepsilon_n\}$ are i.i.d. processes, then the solution $\{Z_n\}$ is strictly stationary and ergodic.

Let $\gamma_i = \alpha_i \beta_i$ for $i = 1,2$ and $B_n(1) = \beta_1(K'_n - \alpha_1)$ and $B_n(2) = \beta_2(K''_n - \alpha_2)$. Then (II.C.1.1) and (II.C.2.1) have the same form. That is (II.C.1.1) is an RCA(2) model if the innovation of NLAR(2) satisfies condition (iii) above. Thus applying the results in [Ref. 16: p.31 and p.37], there exists a unique strictly stationary and ergodic solution to (II.C.2.1) for γ_i and $B_n(i)$ as defined above, if and only if all of the

roots of the characteristic equation

$$(t^2 - \alpha_1\beta_1^2 t - \alpha_2\beta_2^2)(t^2 - \alpha_2\beta_2^2) = 0, \qquad \text{(II.C.2.2)}$$

are within the unit circle, i.e. iff $\alpha_1\beta_1^2 + \alpha_2\beta_2^2 < 1$. This is satisfied for the conditions on the parameters defining NLAR(2), thus establishing the existence of the model (II.C.1.1).

No marginal distribution is ascribed to solutions of the general RCA(p) models in [Ref. 16]. It is, in fact, determined by the independent choices of the innovation and the random coefficients. However, by specifying the marginal distribution and the random coefficients, in NLAR(2) the innovation is restricted more than in the RCA(p) model. If the X_n in (II.C.1.1) or Z_n in (II.C.2.1) have a standard Laplace marginal distribution, then all their moments are given by (II.B.1.3). From (II.C.1.1) or (II.C.2.1), it follows that for all $p = 1,2,\ldots$

$$E(\varepsilon_n^{2k}) = \{(2k)!\}\Big[1 - (\alpha_1\beta_1^{2k} + \alpha_2\beta_2^{2k})$$

$$- \sum_{i=1}^{k-1} \{(\alpha_1\beta_1^{2(k-i)} + \alpha_2\beta_2^{2(k-i)})E(\varepsilon_n^{2(k-i)})/\{(2i)!\}\}\Big] > 0, \qquad \text{(II.C.2.2)}$$

and for this to be true it is necessary that

$$\alpha_1\beta_1^{2k} + \alpha_2\beta_2^{2k} < 1. \qquad \text{(II.C.2.3)}$$

Since α_1 and α_2 are probabilities it is necessary that $|\beta_i| \le 1$ for $i = 1,2$ for (II.C.2.3) to hold. If not, there exists for every $\alpha_1 > 0$ and $\alpha_2 > 0$ an integer m, such that $\alpha_1\beta_1^{2m}$ or $\alpha_2\beta_2^{2m}$ is greater than 1.

We have now established the necessary conditions on the innovation $\{\varepsilon_n\}$, and on β_1 and β_2--namely that $|\beta_i| \le 1$, $i = 1,2$--for the existence of a unique strictly stationary solution to (II.C.2.1) with a marginal Laplace distribution and with the random coefficients given by (II.C.1.2). In the next theorem, we show that $|\beta_i| \le 1$ for $i = 1,2$ is also a sufficient condition and that such an innovation random variable ε_n exists. We also give its explicit form--a convex combination of Laplace random variables. For simplicity, the parameter space is regarded as being described by strict inequalities for

THEOREM 1. Let $\{X_n\}$ be a stationary process with standard Laplace marginal distribution. For all n, let equations (II.C.1.1) and (II.C.1.2) hold with $0 < |\beta_i| < 1$, $0 < \alpha_i < 1$ for $i = 1,2$ and $\alpha_1 + \alpha_2 < 1$. Then

$$\varepsilon_n = K_n L_n = \begin{cases} L_n & \text{w.p.} \quad 1-p_2-p_3, \\ |b_2|L_n & \text{w.p.} \quad p_2, \\ |b_3|L_n & \text{w.p.} \quad p_3, \end{cases} \qquad \text{(II.C.2.4)}$$

where $\{L_n\}$ are i.i.d. standard Laplace variates; the K_n's have values in $\{1, |b_2|, |b_3|\}$ and are independent of $\{L_n\}$ and $\{K_n', K_n''\}$ for all n. They are also independent of $X_{n-1}, X_{n-2}, \ldots$. Furthermore,

$$p_2 = \{(\alpha_1\beta_1^2 + \alpha_2\beta_2^2)b_2^2 - (\alpha_1 + \alpha_2)\beta_1^2\ \beta_2^2\}/(b_2^2 - b_3^2)(1 - b_2^2), \qquad \text{(II.C.2.5)}$$

$$p_3 = \{(\alpha_1 + \alpha_2)\ \beta_1^2\ \beta_2^2 - (\alpha_1\beta_1^2 + \alpha_2\beta_2^2)b_3^2\}/(b_2^2 - b_3^2)(1 - b_3^2), \qquad \text{(II.C.2.6)}$$

$$1 > b_2^2 = \frac{1}{2}\{s + (s^2 - 4r)^{1/2}\} > b_3^2 = \frac{1}{2}\{s - (s^2 - 4r)^{1/2}\} > 0, \qquad \text{(II.C.2.7)}$$

$$s = (1 - \alpha_1)\beta_1^2 + (1 - \alpha_2)\beta_2^2, \text{ and} \qquad \text{(II.C.2.8)}$$

$$r = (1 - \alpha_1 - \alpha_2)\beta_1^2\beta_2^2. \qquad \text{(II.C.2.9)}$$

Proof:

For the NLAR(2) model specified by (II.C.1.1), (II.C.1.2) and (II.C.2.4) - (II.C.2.9), let $\phi_X(\omega)$ and $\phi_\varepsilon(\omega)$ be the characteristic functions of the $\{X_n\}$ and $\{\varepsilon_n\}$ sequences. If $\{X_n\}$ is stationary, then

$$\phi_X(\omega) = \phi_\varepsilon(\omega)\{\alpha_1\phi_X(\beta_1\omega) + \alpha_2\phi_X(\beta_2\omega) + (1-\alpha_1-\alpha_2)\}. \qquad \text{(II.C.2.10)}$$

Assuming a standard Laplace marginal distribution for $\{X_n\}$, the independent distribution of $\{\varepsilon_n\}$ has a characteristic function, possibly improper, given by

$$\phi_\varepsilon(\omega) = \frac{(1+\beta_1^2\omega^2)(1+\beta_2^2\omega^2)}{(1+\omega^2)[(1-\alpha_1-\alpha_2)\beta_1^2\beta_2^2\omega^4 + \{(1-\alpha_1)\beta_1^2 + (1-\alpha_2)\beta_2^2\}\omega^2 + 1]}.$$

(II.C.2.11)

It is convenient to factor the quadratic in ω^2 in the denominator of (II.C.2.11). The roots of this factor are both real and distinct. To see this, note that

$$\{(1-\alpha_1)\beta_1^2 + (1-\alpha_2)\beta_2^2\}^2 - 4(1-\alpha_1-\alpha_2)\beta_1^2\beta_2^2$$
$$= \{(1-\alpha_1)\beta_1^2 - (1-\alpha_2)\beta_2^2\}^2 + 4\alpha_1\alpha_2\beta_1^2\beta_2^2 > 0.$$

The roots are also both negative, which can be seen by noting that the product $r_1 r_2 = 1/(1-\alpha_1\alpha_2)\beta_1^2\beta_2^2$ is positive, but the sum $r_1 + r_2 = -\{(1-\alpha_1)\beta_1^2 + (1-\alpha_2)\beta_2^2\}/(1-\alpha_1-\alpha_2)\beta_1^2\beta_2^2$ is negative.

Letting $r_1 = -1/b_2^2$ and $r_2 = -1/b_3^2$, we can rewrite (II.C.2.11) using partial fraction decomposition to obtain

$$\phi_\varepsilon(\omega) = (1-p_2-p_3)\left(\frac{1}{1+\omega^2}\right) + p_2\left(\frac{1}{1+b_2^2\omega^2}\right) + p_3\left(\frac{1}{1+b_3^2\omega^2}\right). \tag{II.C.2.12}$$

From (II.C.2.11)

$$b_2^2 + b_3^2 = (1-\alpha_1)\beta_1^2 + (1-\alpha_2)\beta_2^2 = s \tag{II.C.2.13}$$

and

$$b_3^2 b_2^2 = (1-\alpha_1-\alpha_2)\beta_1^2\beta_2^2 = r. \tag{II.C.2.14}$$

Comparing (II.C.2.12) and (II.C.2.11) term for term also yields

$$p_2(1-b_2^2) + p_3(1-b_3^2) = \alpha_1\beta^2{}_1 + \alpha_2\beta_2^2 \tag{II.C.2.15}$$

and

$$p_2(1-b_2^2)b_3^2 + p_3(1-b_3^2)b_2^2 = (\alpha_1+\alpha_2)\beta_1^2\beta_2^2. \qquad \text{(II.C.2.16)}$$

Expressions for b_2^2, b_3^2, p_2 and p_3 are obtained in terms of α_1, α_2, β_1 and β_2, by solving (II.C.2.13) - (II.C.2.16). From solving (II.C.2.15) and (II.C.2.16) simultaneously, we obtain (II.C.2.5) and (II.C.2.6). Equations for b_2^2 and b_3^2 given in (II.C.2.7) are obtained from solving (II.C.2.13) and (II.C.2.14) simultaneously. Arbitrarily, let b_2^2 be the larger value.

It remains now to show that the inversion of (II.C.2.12) will, in fact, yield a function that is a probability density and is the mixture of densities for scaled Laplace variables. To do this, we show that p_2 and p_3 can be interpreted as probabilities and that $p_2 + p_3 < 1$.

To establish that $p_2 + p_3 < 1$, we have, after adding (II.C.2.5) and (II.C.2.6)

$$p_2 + p_3 = \frac{(\alpha_1\beta_1^2+\alpha_2\beta_2^2) - (\alpha_1+\alpha_2)\beta_1^2\beta_2^2}{(1-b_2^2)(1-b_3^2)}. \qquad \text{(II.C.2.17)}$$

Multiplying out $(1-b_2^2)(1-b_3^2)$ and using (II.C.2.13) and (II.C.2.14), we have, after some rearrangement

$$p_2 + p_3 = 1 - \frac{(1-\beta_1^2)(1-\beta_2^2)}{(1-\beta_1^2)(1-\beta_2^2) + \alpha_1\beta_1^2(1-\beta_2^2) + \alpha_2\beta_2^2(1-\beta_1^2)} . \qquad \text{(II.C.2.18)}$$

This expression is clearly positive and less than one, from which it follows that $p_2+p_3<1$.

To show that p_2 and p_3 are probabilities, it remains to show that they are both positive. To do this, it is shown that the numerators and the denominators of (II.C.2.5) and (II.C.2.6) are positive. For the denominators, this requires that $0<b_2^2$, $b_3^2<1$, which is shown by noting $0<(1-b_2^2)(1-b_3^2)<1$. From (II.C.2.17) and (II.C.2.18), it follows that

$$(1-b_2^2)(1-b_3^2) = (1-\beta_1^2)(1-\beta_2^2) + \alpha_1\beta_1^2(1-\beta_1^2) + \alpha_2\beta_2^2(1-\beta_1^2) > 0.$$

Also,

$$\begin{aligned} 1 - (1-b_2^2)(1-b_3^2) &= (b_2^2 + b_3^2) - b_2^2 b_3^2 \\ &= (1-\alpha_1)\beta_1^2 + (1-\alpha_2)\beta_2^2 - (1-\alpha_1-\alpha_2)\beta_1^2\beta_2^2 \\ &= (1-\alpha_1)\beta_1^2(1-\beta_1^2) + (1-\alpha_2)\beta_2^2(1-\beta_2^2) + \beta_1^2\beta_2^2 > 0. \end{aligned}$$

Therefore, b_2^2 and b_3^2 are less than one, so p_2 and p_3 have positive denominators.

To see that p_2 and p_3 have positive numerators, note that it must be true that

$$b_3^2 < b = \frac{(\alpha_1+\alpha_2)\beta_1^2\beta_2^2}{(\alpha_1\beta_1^2+\alpha_2\beta_2^2)} < b_2^2 \ . \qquad \text{(II.C.2.19)}$$

Using (II.C.2.8) and (II.C.2.9), (II.C.2.19) is equivalent to

$$-(s^2-4r)^{1/2} < 2b - s < (s^2-4r)^{1/2},$$

or

$$s^2 - 4r > (s-2b)^2,$$

or

$$sb - b^2 - r > 0. \tag{II.C.2.20}$$

But the lefthand side of (II.C.2.20) is

$$\frac{\alpha_1\alpha_2\beta_1^2\beta_2^2(\beta_1^2-\beta_2^2)^2}{(\alpha_1\beta_1^2+\alpha_2\beta_2^2)^2},$$

which is strictly positive.

Therefore, p_2 and p_3 are both positive and $p_2+p_3<1$. Therefore, p_2, p_3 and $1-p_2-p_3$ can be regarded as probabilities. Therefore ε_n has a proper density which can be generated as the mixture of three Laplaces with scale parameters 1, $|b_2|$ and $|b_3|$, respectively. Q.E.D.

The general NLAR(2) model uses the four parameters to achieve a wide range of sample path behavior. Figure II.C.2.1 illustrates four different realizations of the NLAR(2) process. In each case, the theoretical autocorrelations are identical with $\rho(1) = 0.64$ and $\rho(2) = 0.5$. Also, note that each sample path was generated from the same i.i.d. standard Laplace sequence $\{L_n\}$, such that $(X_1,X_2) = (L_1,L_2)$. Since this is not the steady state bivariate distribution of (X_n,X_{n-1}), the sample paths illustrated in Figure II.C.2.1 are displayed beginning

NLAR(2): SAMPLE PATHS; $\rho(1) = .64$ AND $\rho(2) = .5$

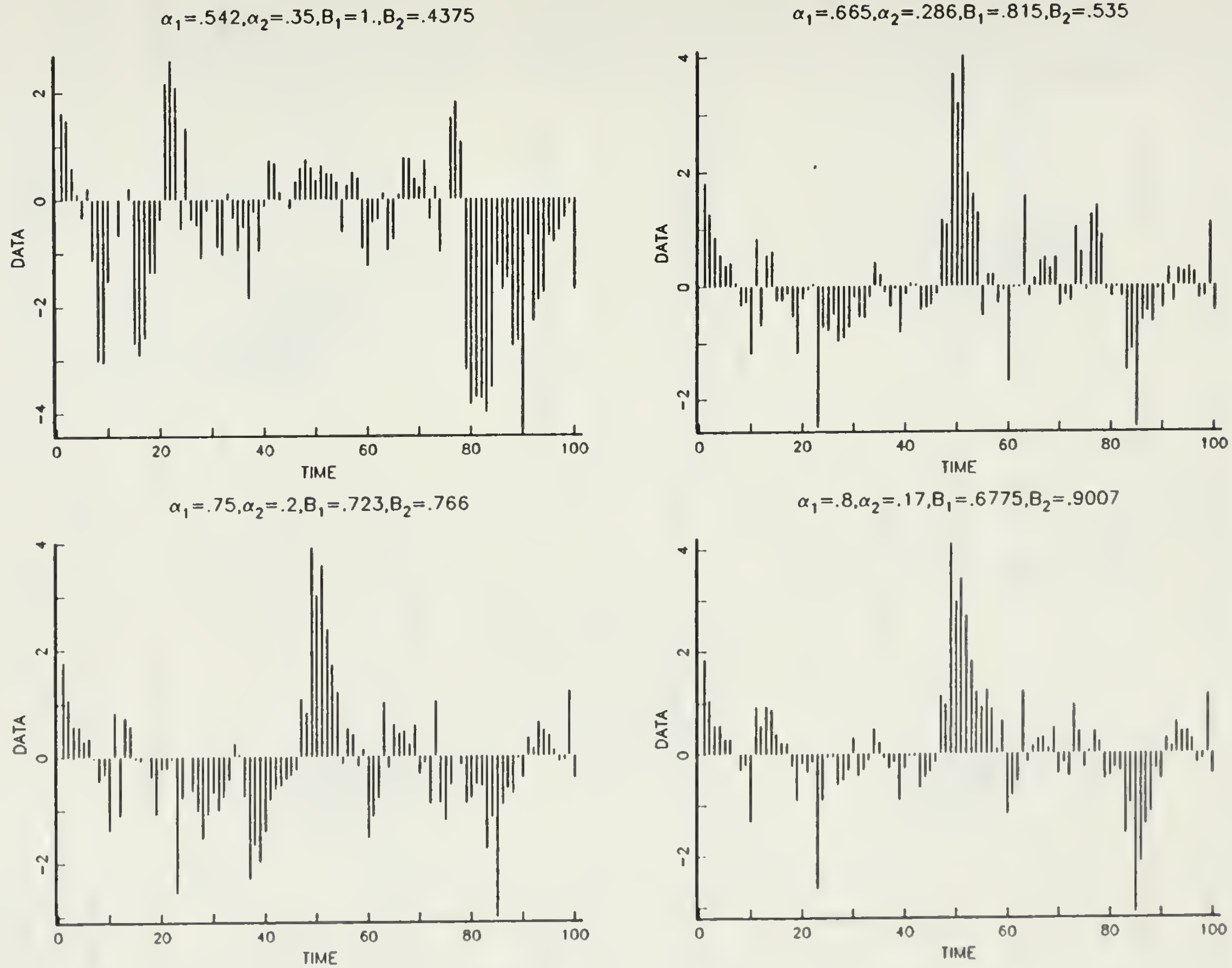


Figure II.C.2.1. NLAR(2): Sample Paths; $\rho(1)=.64$ and $\rho(2)=.5$

with X_{501} to avoid the initial transient behavior of the process. The true value of each parameter is displayed above the corresponding sample path. Figure II.C.2.2 contains the scatter plots for each sample in Figure II.C.2.1. The sample size in each plot is 2000.

Many special cases of the NLAR(2) model could be mentioned. The following have one or more of the parameters at their boundary value and have valid but less complicated results for the distribution of $\{\varepsilon_n\}$ in (II.C.2.4). If $\alpha_1 = \alpha_2 = 0$, then $\{\varepsilon_n\}$ is the i.i.d. sequence $\{L_n\}$ and $X_n = \varepsilon_n$. If $\alpha_1 = 1$ then $\{\varepsilon_n\}$ is the innovation of the LAR(1) model derived from (II.B.1.7) and (II.B.1.8). If $|\beta_1| = |\beta_2| = 1$ and $\alpha_1 + \alpha_2 < 1$ then each ε_n is distributed as a scaled Laplace random variable, $\sqrt{1-\alpha_1-\alpha_2}\, L_n$. These models are called the TLAR(2) models, which are easily extendable to higher-order autoregressions, as discussed in Section II.E. If $\alpha_1 < 1$ and $\alpha_2 = 0$ or $\beta_2 = 0$, then $\{\varepsilon_n\}$ is the innovation of the new first-order autoregressive model NLAR(1). This model is the subject of Section II.D.

3. Autocorrelation Structure

In this section, it is shown that the autocorrelations $\rho(\ell) = \mathrm{Corr}(X_n, X_{n-\ell})$, $\ell = 0, \pm1, \pm2, \ldots$ of the NLAR(2) model satisfy the Yule-Walker type difference equations; thus the second moment dependency aspects are indistinguishable in form from those for the AR(2) process. We also compare the admissible regions of an AR(2) with (i) an NLAR(2) with 4 parameters and (ii) an NLAR(2) with only two parameters.

From the independence of $\{K_n\}$ and $\{K'_n, K''_n\}$, and (II.C.1.1), (II.C.1.2) and (II.C.2.4), we see that $E(K'_n) = \alpha_1$, $E(K''_n) = \alpha_2$ and

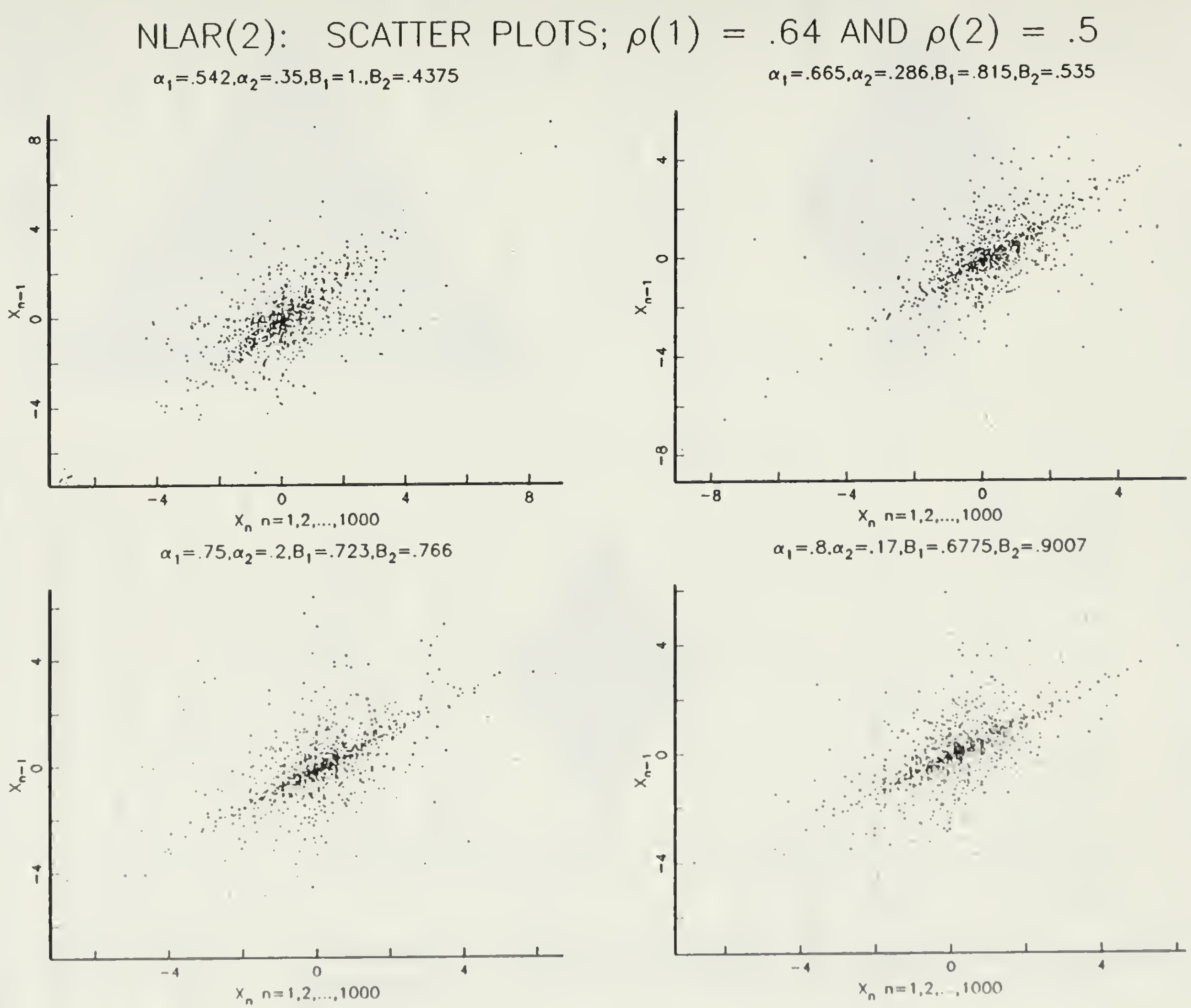


Figure II.C.2.2. NLAR(2): Scatter Plots; $\rho(1)$=.64 and $\rho(2)$=.5

$E(\varepsilon_n) = E(K_n)E(L_n) = 0$. Multiplying (II.C.1.1) on both sides by $X_{n-\ell}$ we have for $\ell \geq 1$, $E(X_n X_{n-\ell}) = \alpha_1\beta_1 E(X_{n-1}X_{n-\ell}) + \alpha_2\beta_2 E(X_{n-2}X_{n-\ell})$. Dividing by $\mathrm{Var}(X_n)$ we have $\rho(-\ell) = \alpha_1\beta_1\rho(\ell - 1) + \alpha_2\beta_2\rho(\ell - 2)$, since $\rho(-\ell) = \rho(\ell)$. Substituting $\alpha_i\beta_i = a_i$ for $i = 1,2$ and $\rho(0) = 1$, we have

$$\rho(1) = a_1 + a_2\rho(1)$$

$$\rho(2) = a_1\rho(1) + a_2, \qquad \text{(II.C.3.1)}$$

which are the same equations as those which occur for the AR(2) process.

Since $|\beta_i| \leq 1$ for $i = 1,2$ and $\alpha_1 + \alpha_2 \leq 1$ in NLAR(2), the usual triangular admissible region for AR(2) given in Box and Jenkins [Ref. 23: p. 61] shrinks to the interior of a diamond-shaped area in $(a_1 = \alpha_1\beta_1,\ a_2 = \alpha_2\beta_2)$ coordinates: $|a_1| + |a_2| \leq 1$. (See Figures II.C.3.1a and 1b). In $(\rho(1),\ \rho(2))$ coordinates the equation $\rho(1)^2 = (1 + \rho(2))/2$ defining allowable combinations of $\rho(1)$ and $\rho(2)$ in AR(2) also changes. For NLAR(2), the space in $(\rho(1),\ \rho(2))$ coordinates becomes a triangular region bounded below by $|\rho(1)| = \frac{1}{2}\{1 + \rho(2)\}$. (See Figures II.C.3.2a and 2b).

The reduction in allowable parameter or correlation combinations for NLAR(2) over the AR(2) model is not large. This encouraged us to consider a 2-parameter NLAR(2) model by specifying $\alpha_i = \beta_i^2$, for $i = 1,2$, so that $a_i = \beta_i^3$. The parameter space in (a_1,a_2) coordinates becomes the symmetric region bounded by the curves $\beta_2^3 = \pm\ (1 - \beta_1^2)^{3/2}$ (see Figure II.C.3.1c). In $(\beta_1,\ \beta_2)$ coordinates the admissible region of the two parameter model is bounded by the unit circle $\beta_1^2 + \beta_2^2 = 1$. Using only

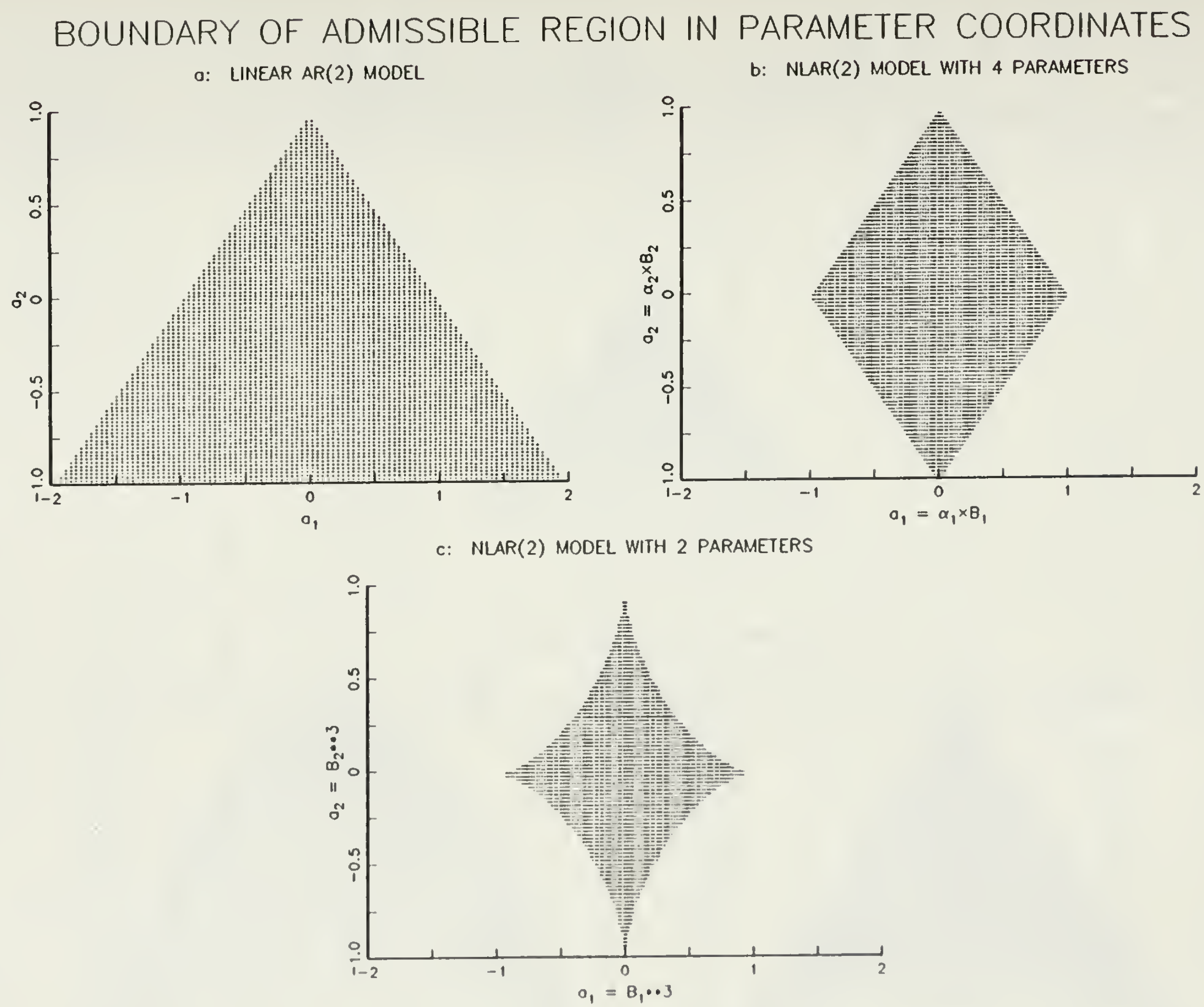


Figure II.C.3.1. Boundary of Admissible Region in Parameter Coordinates for Linear AR(2) and NLAR(2) Processes

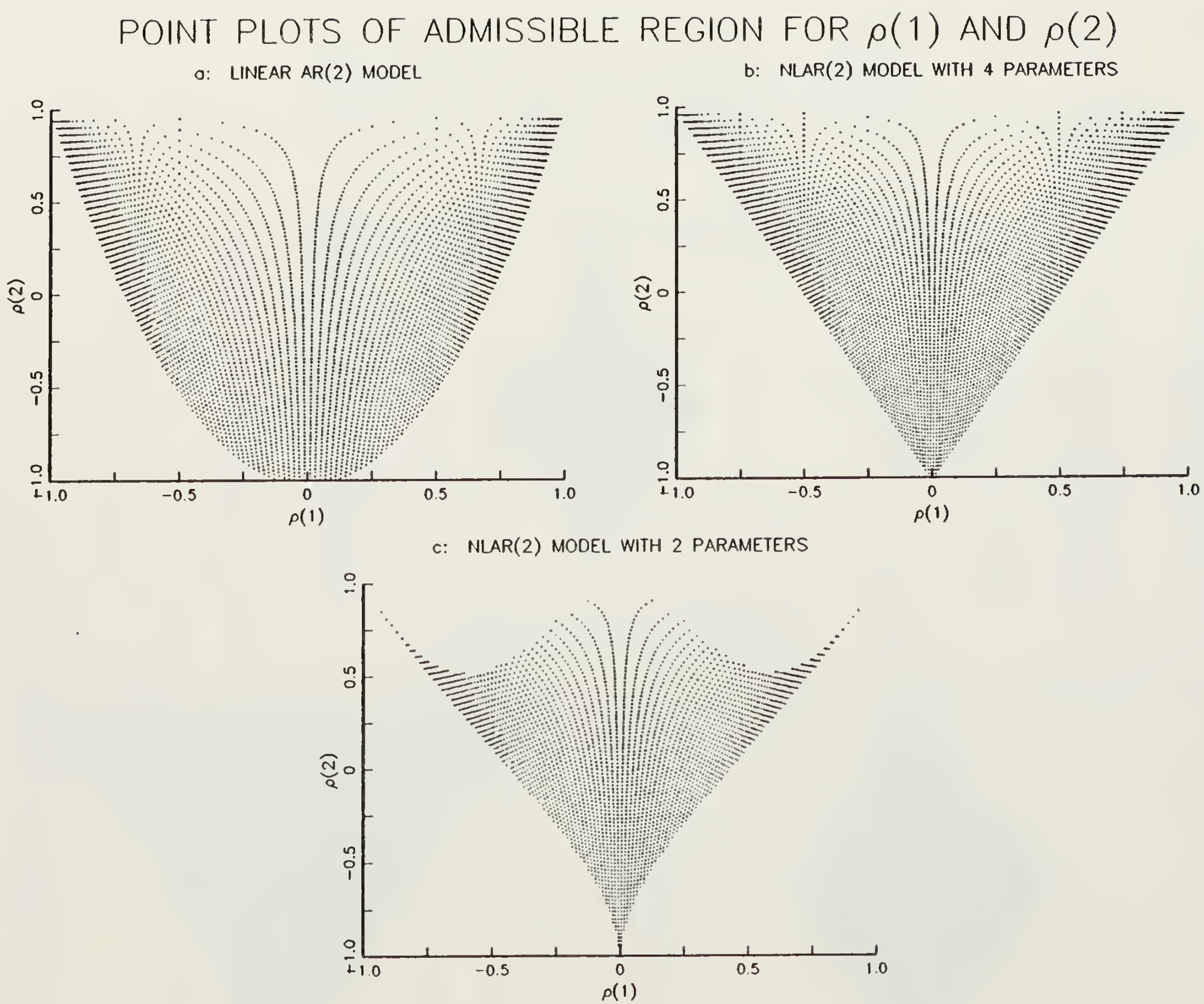


Figure II.C.3.2. Point Plots of Admissible Region for ρ(1) and ρ(2) for Linear AR(2) and NLAR(2) Processes

two parameters leads to the admissible region in Figure II.C.3.2c for $(\rho(1), \rho(2))$ space. The $(\rho(1), \rho(2))$ space was obtained by transforming the lines $\beta_2^3 = a_2 = c$, $-1 \leq c \leq 1$, in Figure II.C.3.1c to $\rho(2) = (1-a_2)\rho(1)^2+a_2$, where $|\rho(1)| \leq a_1/(1-a_2) = \beta_1^3/(1-\beta_2^3)$ and $\beta_2^3 = (1-\beta_1^2)^{3/2}$ if $a_2 \geq 0$ and $\beta_2^3 = -(1-\beta_1^2)^{3/2}$ if $a_2 < 0$.

All the plots in Fig. II.C.3.1 were generated from a grid of equally spaced values of a_1 and a_2. In Fig. II.C.3.1a the points satisfy the Yule-Walker equations (5.1). In Figs. II.C.3.1b and 1c, the points also satisfy the conditions of Theorem 1. In Fig. II.C.3.2 the feasible combinations of $\rho(1)$ and $\rho(2)$ are plotted for those values of a_1 and a_2 from Fig. II.C.3.1 using the Yule-Walker equations (5.1).

4. Directional Moments and Partial Time Reversibility

In the last section, we demonstrated that the second moment dependency aspects of the NLAR(2) model were indistinguishable in form from those of the ordinary AR(2) model. Also, it is well known that if the linear autoregressive model is not Gaussian, then the process is not completely determined by the first and second moments. Thus in model identification it becomes necessary to examine third order moments to further identify the process. Special third order moments $E(X_n^2 X_{n+\ell})$, for all ℓ, are known as directional moments. If the directional moments for all ℓ are equal, which is necessary for a process to be fully time reversible, we say the process is partially time reversible in the sense of directional moments.

A process is fully time reversible (Lawrance [Ref. 27]) if the joint distribution of $X_n, X_{n+1}, \ldots, X_{n+r}$, is the same as that for $X_{n+r}, X_{n+r-1}, \ldots, X_n$ for all r and for all n. Since LAR(1), a special case of

NLAR(2), is not fully time reversible, NLAR(2) is in general not time reversible.

In this section we show by induction arguments that all the third order moments of NLAR(2) are the same as those for Gaussian AR(2) model; i.e. $E(X_i X_j X_k) = 0$ for i, j, k. This implies particularly that the directional moments of NLAR(2) are equal and therefore that NLAR(2) is always partially time reversible.

In Section II.B, we found that $E(X_i^3) = 0$ for all i since X_i is marginally Standard Laplace. It is easy to establish the following two equations:

$$E(X_n X_{n-1}^2) = \beta_2 \alpha_2 E(X_n^2 X_{n-1}); \tag{II.C.4.1}$$

$$E(X_n^2 X_{n-1}) = \{(\beta_2^2 \alpha_2)/(1 - 2\beta_1 \beta_2 \alpha_1 \alpha_2)\} E(X_n X_{n-1}^2). \tag{II.C.4.2}$$

Solving (II.C.4.1) and (II.C.4.2), simultaneously yields $E(X_n X_{n-1}^2) = E(X_n^2 X_{n-1}) = 0$.

Now, using separate induction arguments and the stationarity assumption, we establish that $E(X_n X_{n-\ell}^2) = 0$ for all $\ell \geqq 1$, and $E(X_n^2 X_{n-k}) = 0$ for all $k \geqq 1$.

The proof of $E(X_n X_{n-\ell}^2) = 0$ is straightforward.

To prove $E(X_n^2 X_{n-k}) = 0$, we first show that the expectation of special third order moments of the form $X_n X_{n-1} X_{n-k}$ for $k \geqq 2$ is zero. Define $\mu_k = E(X_n X_{n-1} X_{n-k})$ and assume $E(X_n^2 X_{n-j}) = 0$, $j \leqq k - 1$. From (II.C.1.1),

$$\mu_k = E(X_n X_{n-1} X_{n-k}) = \alpha_1\beta_1 E(X_n^2 X_{n-(k-1)}) + \alpha_2\beta_2 E(X_n X_{n-1} X_{n-(k-1)})$$

$$= \alpha_2\beta_2\mu_{k-1} = \ldots = (\alpha_2\beta_2)^{k-1}\mu_1. \qquad \text{(II.C.4.3)}$$

Now from (II.C.4.1) and (II.C.4.2), we have

$$\mu_1 = E(X_n X_{n-1}^2) = \alpha_2\beta_2 E(X_n^2 X_{n-1}) = 0. \quad \text{Therefore } \mu_k = 0.$$

We now proceed to show that $E(X_i X_j X_k) = 0$ for all i, j, k. Without loss of generality let $i < j < k$ so that $k = i + n$, $j = i + m$ and $n > m$. Therefore by stationarity $E(X_i X_j X_k) = E(X_i X_{i+m} X_{i+n}) = E(X_i X_{i-(n-m)} X_{i-n})$. Fixing m so that $0 < m < n$ we use induction on n. Let $n = 2$, implying $m = 1$. The first step in the induction follows from $E(X_i X_{i-1} X_{i-2}) = \mu_2 = 0$. Next assume that for $m < n \leq K$, $E(X_i X_{i-(n-m)} X_{i-n}) = 0$. Now we show that $E(X_i X_{i-(K+1-m)} X_{i-(K+1)}) = 0$. Using (II.C.1.1), we write

$$\begin{aligned} E(X_i X_{i-(K+1-m)} X_{i-(K+1)}) &= \alpha_1\beta_1 E(X_{i-1} X_{i-(K+1-m)} X_{i-(K+1)}) \\ &\quad + \alpha_2\beta_2 E(X_{i-2} X_{i-(K+1-m)} X_{i-(K+1)}) \\ &\quad + E(\varepsilon_i X_{i-(K+1-m)} X_{i-(K+1)}). \end{aligned}$$

Now $E(\varepsilon_i X_{i-(K+1-m)} X_{i-(K+1)}) = E(\varepsilon_i) E(X_{i-(K+1-m)} X_{i-(K+1)}) = 0$ and $E(X_{i-1} X_{i-(K+1-m)} X_{i-(K+1)}) = E(X_i X_{i-(K-m)} X_{i-K}) = 0$ by stationarity and the assumption. Likewise $E(X_{i-2} X_{i-(K+1-m)} X_{i-(K+1)}) = E(X_i X_{i-\{(K-1)-m\}} X_{i-(K-1)}) = 0$. This completes the induction.

An immediate result from the argument about third moments is that $Z_n = X_n - X_{n-1}$ for $\{X_n\}$ of the NLAR(2) is not skewed.

The residual analysis in [Ref. 6] and [Ref. 22] using cross correlations between linear autoregressive residuals $R_n = X_n - a_1X_{n-1} - a_2X_{n-2}$, and their squares R_n^2, does not shed any new light on the directionality/reversibility in the NLAR(2) model or help in identifying the appropriateness of the Laplacian model. This is because all third moments have zero expectation. Thus, we see that $E(R_n^2R_{n+\ell}) = E(R_nR^2_{n+\ell}) = 0$ for all ℓ.

Note that the basis for the residual analysis using the $\{R_n\}$ process is that this process is uncorrelated but not necessarily independent. The moment results show that the R_n's have zero skewness. In fact, it is easy to show that the distribution of R_n is the same as the distribution of $-R_n$. Thus the R_n's are symmetric although they will, of course, not have Laplacian distributions.

In Chapter IV of this thesis, a residual analysis based on certain fourth-order moments is presented.

D. THE NEW LAPLACE FIRST-ORDER AUTOREGRESSIVE MODEL, NLAR(1)

1. Introduction

The new Laplace first-order autoregressive model is another special case of the NLARMA(p,q) model when q=0 and p=1. This is, of course, a special case of the NLAR(2) model, where either α_2 and/or β_2 are zero in (II.C.1.1). Examples of the different sample path behavior obtainable from the NLAR(1) Process are given in Figure II.D.1.1. Note that each sample has the same value of lag-1 serial correlation, i.e. $\rho(1) = \mathrm{Corr}(X_n, X_{n-1})$. In Figure II.D.1.2 are the corresponding scatter plots for the samples in Figure II.D.1.1. In the scatter plot labeled, "α_1=1", the distinctive regression line, $x_n = \rho x_{n-1}$ is clearly visible

NLAR(1): SAMPLE PATHS; $\rho(1) = .64$

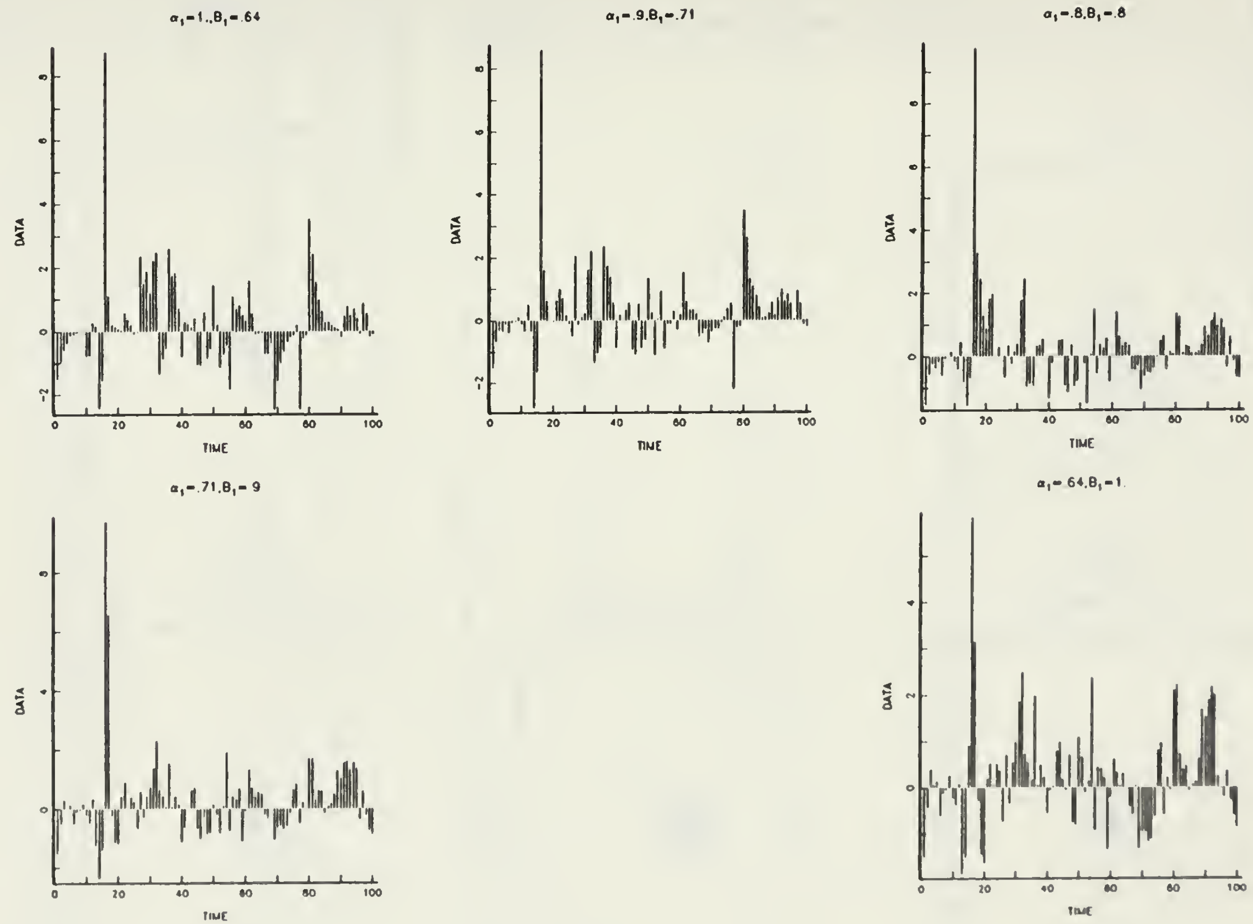


Figure II.D.1.1. NLAR(1): Sample Paths; $\rho(1)=.64$

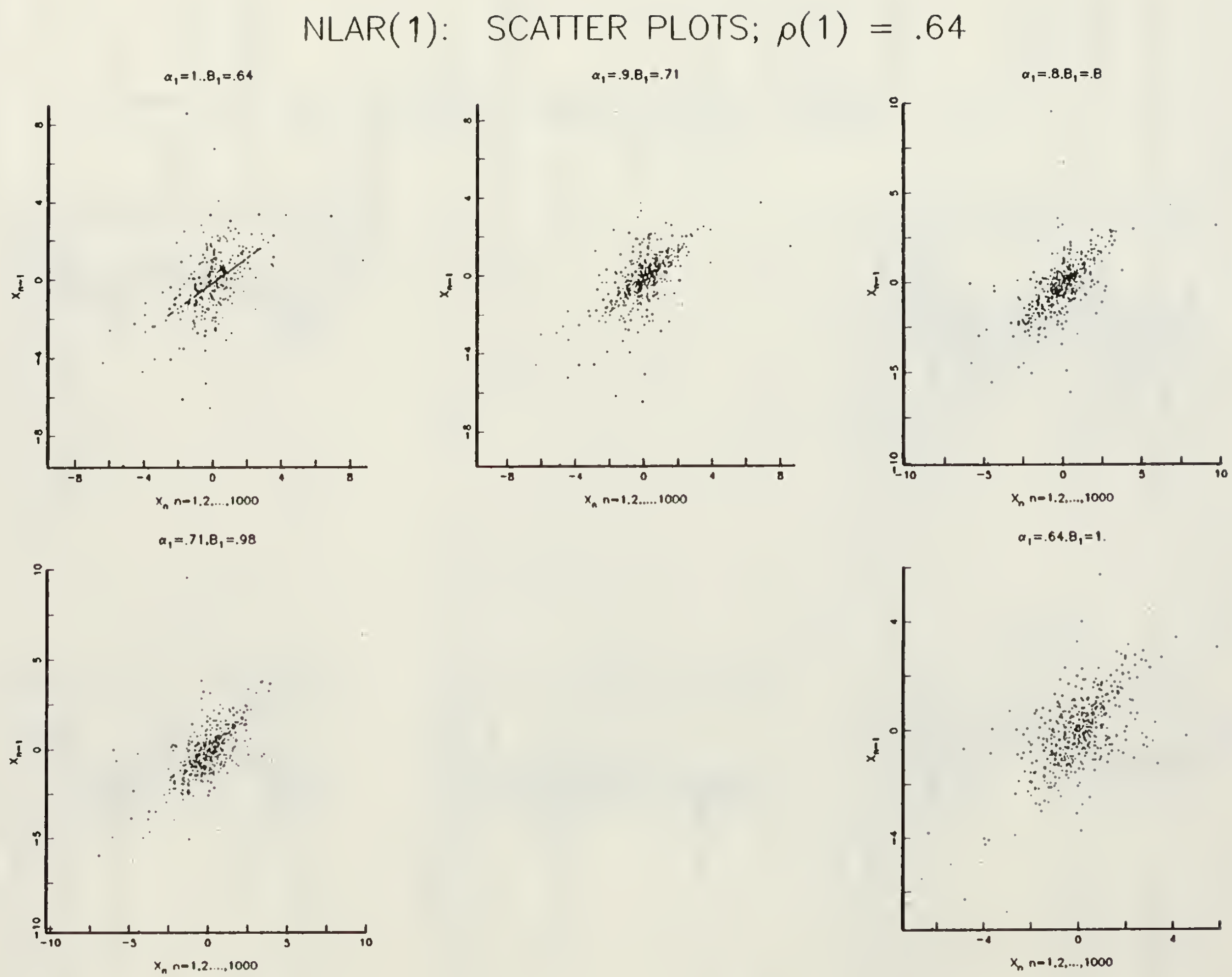


Figure II.D.1.2. NLAR(1): Scatter Plots; $\rho(1)=.64$

for the LAR(1) process. This is produced as explained in Section II.B, because the innovation, ε_n, can be zero with non-zero probability.

The two-parameter autoregressive model generates an $\{X_n\}$ sequence which satisfies

$$X_n = K'_n \beta_1 X_{n-1} + \varepsilon_n, \qquad \text{(II.D.1.1)}$$
$$= K'_n \beta_1 X_{n-1} + K_n L_n$$

where

$$K'_n = \begin{cases} 1 & \text{w.p. } \alpha_1 \\ 0 & \text{w.p. } 1-\alpha_1 \end{cases}; \qquad K_n = \begin{cases} 1 & \text{w.p. } 1-p_2 \\ \sqrt{1-\alpha_1}\,|\beta_1|\,L_n & \text{w.p. } p_2 \end{cases} \qquad \text{(II.D.1.2)}$$

and

$$p_2 = \alpha_1 \beta_1^2 / \{1 - (1-\alpha_1)\beta_1^2\}. \qquad \text{(II.D.1.3)}$$

Also, $\{K'_n\}$, $\{K_n\}$, $\{L_n\}$ are i.i.d. sequences independent of each other and independent of $X_{n-1}, X_{n-2}, \ldots$.

From (II.D.1.2) and (II.D.1.3), we see that the inversion of the characteristic function for ε_n, letting $\lambda = (1-\alpha_1)^{-1/2}(|\beta_1|)^{-1}$, gives for $0<\alpha_1<1$

$$f_{\varepsilon_n}(x) = \frac{(1-p_2)}{2} e^{-|x|} + \frac{\lambda p_2}{2} e^{-\lambda|x|}, \qquad \text{(II.D.1.4)}$$

which is a convex mixture of Laplace densities. This result also follows directly from Section II.C.3, since the NLAR(1) model is an

NLAR(2) model. Likewise, the correlation structure and partial time reversibility in the sense of directional moments are the corresponding results for the NLAR(2) model with $\alpha_2=0$ or $\beta_2=0$. That is

$$\mathrm{Corr}(X_n, X_{n-k}) = (\alpha\beta)^{|k|} \quad \text{for all } k = 0, \pm 1, \pm 2, \ldots \qquad \text{(II.D.1.5)}$$

and

$$E(X_n^2 X_{n+k}) = E(X_n X_{n+k}^2) = 0 \quad \text{for all } k = 0, \pm 1, \pm 2. \qquad \text{(II.D.1.6)}$$

We can rewrite (II.D.1.1) as

$$X_n = \varepsilon_n + \sum_{j=1}^{n} \beta_1^j \left(\prod_{i=0}^{j-1} K'_{n-i} \right) \varepsilon_{n-j}. \qquad \text{(II.D.1.7)}$$

From (II.D.1.1), it is clear that X_n depends only on X_{n-1} and ε_n. From (II.D.1.7), we see that X_{n-1} is independent of ε_{n+k} for all $k \geq 0$. Hence $\{X_n\}$ is a first-order Markov process and starting X_0 with a standard Laplace distribution makes $\{X_n\}$ stationary.

The remainder of this section is devoted to specific results for the NLAR(1) process which have not been shown in the more general NLAR(2) model. The extension of these results to the NLAR(2) process would require the joint distribution of $\{X_n, X_{n-1}, X_{n-2}\}$, which has not been derived. The conditional density of X_n given X_{n-1} is derived, as well as an expression for the joint distribution of the X_n. The distribution for the differences $Z_n=X_n-X_{n-1}$ is also derived. Parameter estimation is discussed in the context of moment estimators and least

squares using the linearized residual. The problems with finding the maximum likelihood estimators of α_1 and β_1 are also addressed.

2. Conditional Density and the Joint Density of $(X_n,\ldots,X_1)$

To find the conditional density of X_n, given X_{n-1}, we use (II.D.1.1) - (II.D.1.4) to evaluate $P(X_n<x_n|X_{n-1})$. We have for $\alpha_1<1$, which eliminates the LAR(1) process,

$$
\begin{aligned}
P(X_n<x_n|X_{n-1}) &= P(K_n'\beta_1 X_{n-1} + \varepsilon_n<x_n|X_{n-1}) \\
&= \alpha_1 P(\varepsilon_n<x_n-\beta_1 x_{n-1}) + (1-\alpha_1)P(\varepsilon_n<x_n) \\
&= \alpha_1 \int_{-\infty}^{x_n-\beta_1 x_{n-1}} f_{\varepsilon_n}(x)dx + (1-\alpha_1)\int_{-\infty}^{x_n} f_{\varepsilon_n}(x)dx. \qquad \text{(II.D.2.1)}
\end{aligned}
$$

Differentiating (II.D.2.1) with respect to x_n yields the following expression for $\alpha_1<1$,

$$
f_{X_n|X_{n-1}}(x_n|x_{n-1}) = \alpha_1 f_{\varepsilon_n}(x_n-\beta_1 x_{n-1}) + (1-\alpha_1)f_{\varepsilon_n}(x_n). \qquad \text{(II.D.2.2)}
$$

Examples of (II.D.2.2) for a fixed x_{n-1} and fixed $\gamma = \alpha_1\beta_1 = .64$ are given in Figure II.D.2.1.

Now we can write the joint density $f_{X_n X_{n-1}}(x_n, x_{n-1})$ as the product $f_{X_n|X_{n-1}}(x_n|x_{n-1})f_{X_{n-1}}(x_{n-1})$. In fact, the n-dimensional distribution of $X_n,\ldots,X_1$ is obtained using this product recursively to

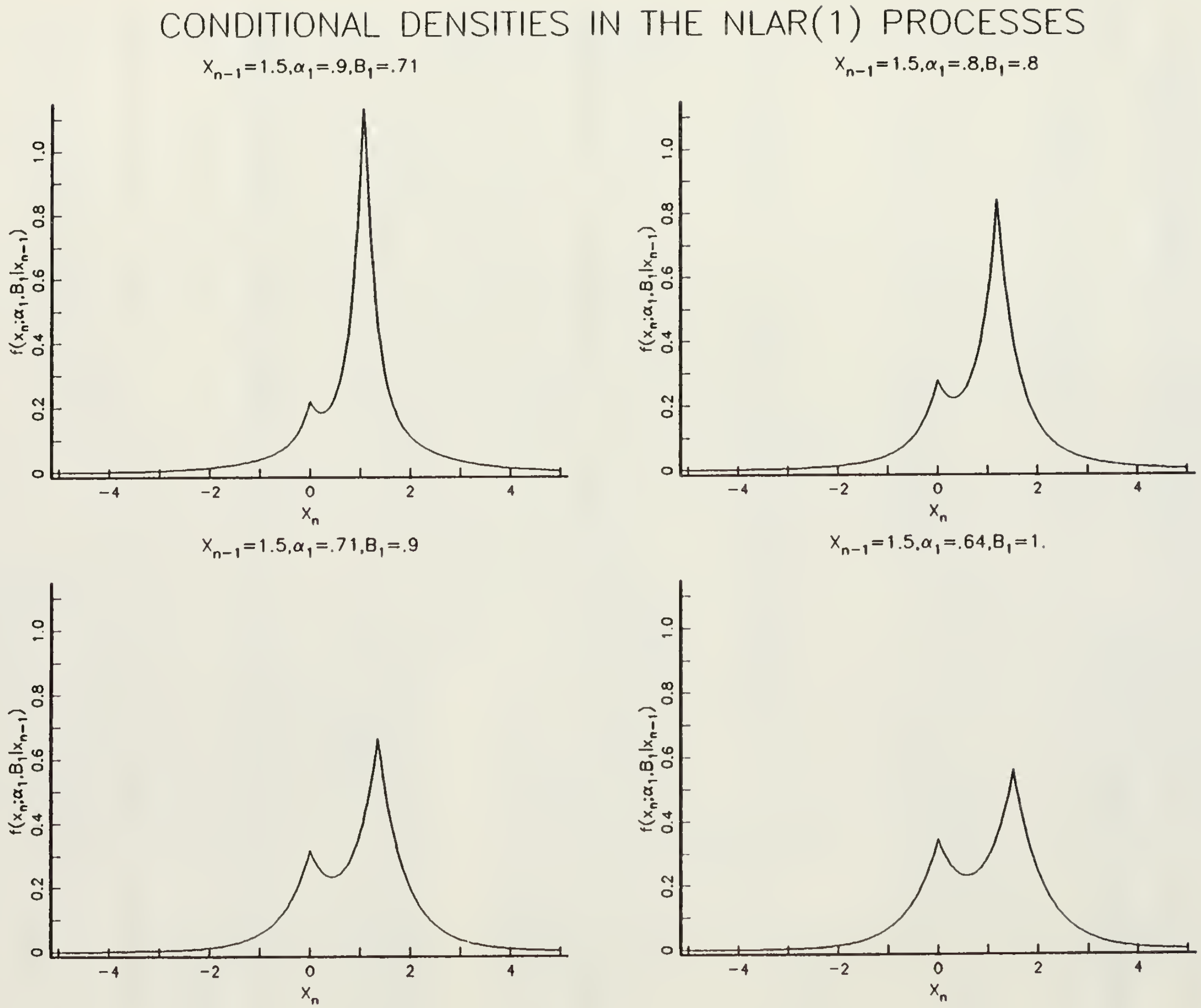


Figure II.D.2.1. Examples of Conditional Density of $X_n|X_{n-1}$ in the NLAR(1) Process for $\alpha_1 < 1$, $|\beta_1| < 1$, and $\alpha_1\beta_1 = .64$

obtain the density

$$f_{X_n \ldots X_1}(x_n,\ldots,x_1) = f_{X_n|X_{n-1}}(x_n|x_{n-1})\, f_{X_{n-1}|X_{n-2}}(x_{n-1}|x_{n-2}) \ldots$$

$$f_{X_2|X_1}(x_2|x_1)\, f_{X_1}(x_1). \tag{II.D.2.4}$$

3. Distribution of Differences and $P(X_{n-1}>X_n)$

We now consider the distribution of the difference $Z_n = X_n - X_{n-1}$. Using (II.D.1.1) - (II.D.1.4) and the fact that ε_n is a convex mixture of Laplacian random variables, we used partial fraction decomposition to invert the characteristic function of Z_n to obtain the following expression for the density:

$$f_{Z_n}(y) = \exp\{-|y|/(1-\beta_1)\}\left\{\frac{\alpha_1(1-\beta_1)}{2}\right\}\left[\frac{p_2}{\{(1-\beta_1)^2-\sigma^2\}} - \frac{(1-p_2)}{\beta_1(2-\beta_1)}\right]$$

$$+ \exp(-|y|/\sigma)(\sigma p_2/2)\left\{\frac{\alpha_1}{\sigma^2 - (1-\beta_1)^2} - \frac{(1-\alpha_1)}{1-\sigma^2}\right\}$$

$$+ \frac{1}{2}\exp(-|y|)\left\{\frac{(1-\alpha_1)p_2}{1-\sigma^2} + \frac{(1-\alpha_1)(1-p_2)}{2} + \frac{\alpha_1(1-p_2)}{\beta_1(2-\beta_1)}\right\}$$

$$+ (1-p_2)(1-\alpha_1)|y|\exp(-|y|)/4, \tag{II.D.3.1}$$

where $\sigma^2 = (1-\alpha_1)\beta_1^2$.

One immediate result is that $f_{Z_n}(y)$ is symmetric about zero and therefore, $P(Z_n<0) = P(Z_n>0) = 1/2$. This demonstrates one additional feature of the partial time reversibility of the NLAR(1) models; i.e., probabilities of a run down $(X_n>X_{n-1})$ and a run up $(X_n<X_{n-1})$ are the same. To evaluate probabilities of higher order runs would require the joint distribution of the sequence $\{Z_n\}$. This result has not been obtained for the NLAR(2) model.

4. Estimation of Serial Correlation

a. Introduction

The purpose of this section is to present estimators of the two parameters α_1 and β_1 whose product is the correlation coefficient in the NLAR(1) models. We assume throughout this section, unless otherwise stated, that $\{X_n\}$ has a standard Laplace $(\mu=0, \lambda=1)$ marginal distribution. Estimation of μ and λ for models that have marginal Laplace distributions are discussed in Chapter III. We also only consider the random coefficient models of the NLAR(1) process, i.e. $\alpha<1$, thus eliminating the LAR(1) model. As was shown in the introduction to this chapter, for $\alpha_1=1$, β_1 can be estimated very efficiently, thus eliminating the need for further discussion.

The method of moments is used first to find an estimator of $\gamma = \alpha_1\beta_1$. The joint moment estimators of α_1 and β_1 are calculated from fourth-order moments. These estimators are used later in an iterative procedure to obtain the joint least squares estimators of α_1 and β_1.

A least squares estimation procedure is defined for the NLAR(1) models using the usual linear residual $R_n = X_n-\alpha_1\beta_1X_{n-1}$.

Minimizing the sum of R_n^2 leads to the usual estimator of γ as given in standard texts on time series. In order to estimate α_1 and β_1 individually, we minimize the square of a particular function of R_n with respect to α_1 and β_1.

In the last part of this section, the problems of maximum likelihood estimation in the NLAR(1) process are discussed. Although no results are presented for the general model, the maximum likelihood estimator of the correlation coefficient in the TLAR(1) model is given.

b. Method of Moments

(1) Estimation of γ by Second-Order Moments. Since X_n is assumed to have a standard Laplace distribution with $E(X_n) = 0$ and $Var(X_n) = 2$, an immediately obvious choice for estimating $\gamma = Corr(X_n, X_{n-1})$ is the following product moment:

$$\hat{\gamma} = \frac{\frac{1}{2} \sum_{i=2}^{n} X_i X_{i-1}}{(n-1)}. \qquad \text{(II.D.4.1)}$$

Taking the expectation of $\hat{\gamma}$ and using (II.D.1.1), we have

$$E(\hat{\gamma}) = \frac{1}{2(n-1)} \sum_{i=2}^{n} E(X_i X_{i-1}) = \frac{1}{2(n-1)} \sum_{i=2}^{n} 2\alpha_1\beta_1 = \alpha_1\beta_1 = \gamma, \qquad \text{(II.D.4.2)}$$

so that the estimator is unbiased.

(2) Joint Estimation of α_1 and β_1 by Fourth-Order Moments. The expectation of fourth-order moments can be calculated using (II.D.1.1) and the fact that $\{X_n\}$ is a stationary process. For example

$$E(X_i^3 X_{i-1}) = 12\alpha_1\beta_1\{1 + (2-\alpha_1)\beta_1^2\}\ , \qquad \text{(II.D.4.3)}$$

$$E(X_i^2 X_{i-1}^2) = 4(1+5\alpha_1\beta_1^2)\ , \qquad \text{(II.D.4.4)}$$

$$E(X_i X_{i-1}^3) = 24\alpha_1\beta_1\ , \qquad \text{(II.D.4.5)}$$

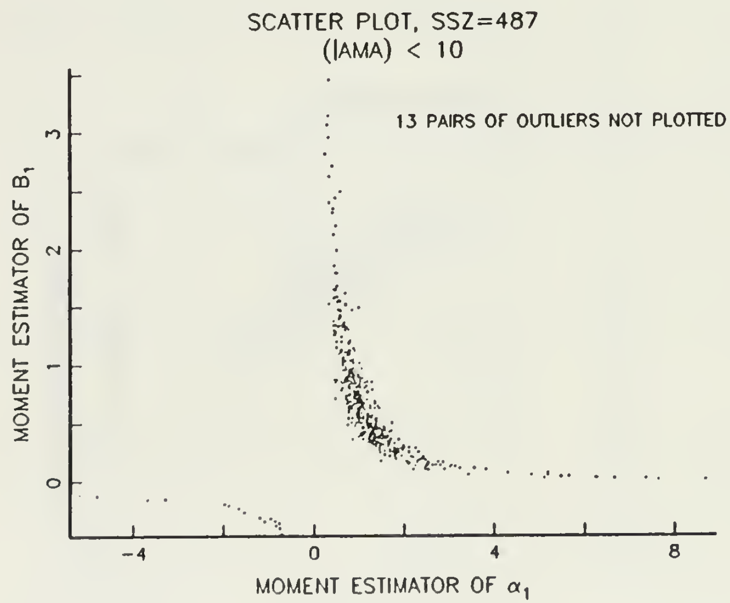
$$E(X_i^2 X_{i-1} X_{i-2}) = 4\alpha_1\beta_1\{1+2\alpha_1\beta_1^2+3\alpha_1(2-\alpha_1)\beta_1^4\}. \qquad \text{(II.D.4.6)}$$

Solving for α_1 and β_1 in different pairs of these equations gives the estimators based on fourth-order moments. It is to our advantage to use the expressions with the lower order moments where possible. Therefore, using $E(X_i^2 X_{i-1}^2)$ and $E(X_i X_{i-1})$ instead of $E(X_i X_{i-1}^3)$, we solve for the following expressions for the joint moment estimators of α_1 and β_1

$$\hat{\alpha}_1 = \frac{5\left\{\sum_{i=2}^{n} x_i x_{i-1}\right\}^2}{(n-1)\left\{\sum_{i=2}^{n} x_i^2 x_{i-1}^2 - 4(n-1)\right\}}\ , \qquad \text{(II.D.4.7)}$$

$$\hat{\beta}_1 = \frac{\sum_{i=2}^{n}(x_1^2 x_{i-1}^2) - 4(n-1)}{10\sum_{i=2}^{n} x_i x_{i-1}}\ . \qquad \text{(II.D.4.8)}$$

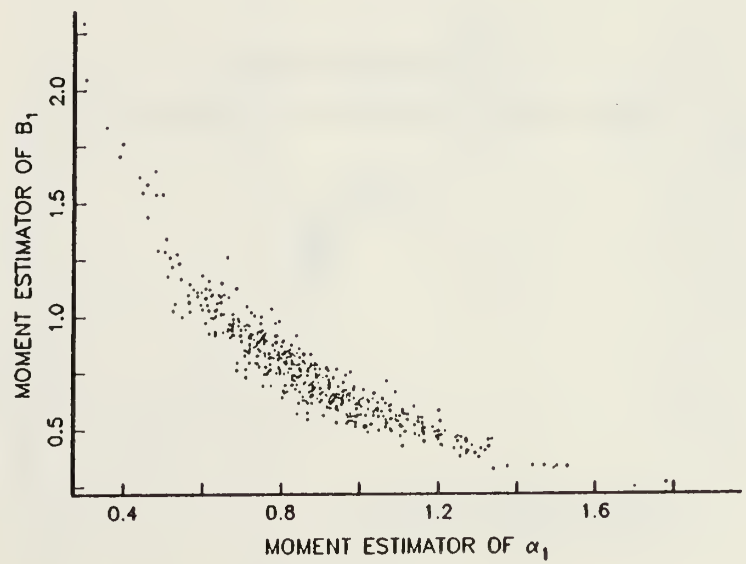
From the scatter plot analyses in Figures II.D.4.1 and II.D.4.2, we see an example of the behavior of this pair of estimators when $\alpha_1 = \beta_1 = .8$ in the NLAR(1) model. Both scatter plots contain 500 pairs (α_1, β_1) derived first from samples of size 250 and then from



SCATTER PLOT TABLE

X	:AMA
Y	:BMA
SELECTION	:(|AMA) < 10
X LABEL	:MOMENT ESTIMATOR OF α_1
Y LABEL	:MOMENT ESTIMATOR OF B_1
NO. OF ELEMENTS	:487
CORRELATION XY	:¯0.30306
RK CORRELATION	:¯0.74097 T=¯24.5
X MEAN	:1.2095
STD. DEVIATION	:1.1755
5-PERCENTILE	:0.39782
25-PERCENTILE	:0.76136
MEDIAN	:1.0576
75-PERCENTILE	:1.4463
95-PERCENTILE	:2.6993
X MIN.	:¯5.1952 ¯4.8067 ¯3.6858
X MAX.	:8.6936 7.6373 7.3554
Y MEAN	:0.69921
STD. DEVIATION	:0.53507
5-PERCENTILE	:0.08003
25-PERCENTILE	:0.37952
MEDIAN	:0.58674
75-PERCENTILE	:0.9056
95-PERCENTILE	:1.6127
Y MIN	:¯0.40734 ¯0.36173 ¯0.34014
Y MAX	:3.4943 3.1861 3.1049

Figure II.D.4.1. Scatter Plot Analysis of Joint Moment Estimators of (α_1, β_1) in the NLAR(1) Process for 500 Samples of Size 250 with $\alpha_1=\beta_1=.8$



SCATTER PLOT TABLE

X	:AMB
Y	:BMB
SELECTION	:ALL
X LABEL	:MOMENT ESTIMATOR OF α_1
Y LABEL	:MOMENT ESTIMATOR OF B_1
NO. OF ELEMENTS	:500
CORRELATION XY	:⁻0.88849
RK CORRELATION	:⁻0.9485/ T=⁻66.865
X MEAN	:0.88332
STD. DEVIATION	:0.21792
5-PERCENTILE	:0.56394
25-PERCENTILE	:0.74219
MEDIAN	:0.86622
75-PERCENTILE	:0.99793
95-PERCENTILE	:1.2589
X MIN.	:0.29594 0.30221 0.35484
X MAX.	:1.9476 1.7804 1.6997
Y MEAN	:0.77997
STD. DEVIATION	:0.2527
5-PERCENTILE	:0.45017
25-PERCENTILE	:0.61127
MEDIAN	:0.75311
75-PERCENTILE	:0.90763
95-PERCENTILE	:1.1657
Y MIN	:0.24398 0.26722 0.28427
Y MAX	:2.3165 2.0686 1.8548

Figure II.D.4.2. Scatter Plot Analysis of Joint Moment Estimators of (α_1, β_1) in the NLAR(1) Process for 500 Samples of Size 2500 with $\alpha_1=\beta_1=.8$

samples of size 2500. It is clear from the equations (II.D.4.7) and (II.D.4.8) that $\hat{\gamma} = \hat{\alpha}_1\hat{\beta}_1$. The hyperbola can be seen in both scatter plots. Both parts are visible for sample size of 250. However, for pairs derived from samples of size 2500, only the part in the first quadrant is visible.

From the Normal probability plots in Figure II.D.4.3, there is little evidence of non-Normality for $\hat{\gamma} = \hat{\alpha}_1\hat{\beta}_1$ for $N = 250$, and less for the estimator derived from samples of size 2500. However, individual estimators $\hat{\alpha}_1$ and $\hat{\beta}_1$ look far less Normal for both sets of sample sizes.

c. Least Squares Estimation in the NLAR(1) Process

(1) The Linear Residual. The properties of the linear residual are developed for use in deriving the least squares estimators of $\gamma = \alpha_1\beta_1$ and for α_1 and β_1 jointly. We begin by rewriting (II.D.1.1) in the RCA(1) form as given in (II.C.2.1). We have

$$X_n = \alpha_1\beta_1X_{n-1} + \beta_1(K_n'-\alpha_1)X_{n-1} + \varepsilon_n. \qquad \text{(II.D.4.9)}$$

From this expression, there are clearly two ways to write down the linear residual, R_n. The usual one from linear theory is, of course

$$R_n = X_n-\alpha_1\beta_1X_{n-1}. \qquad \text{(II.D.4.10)}$$

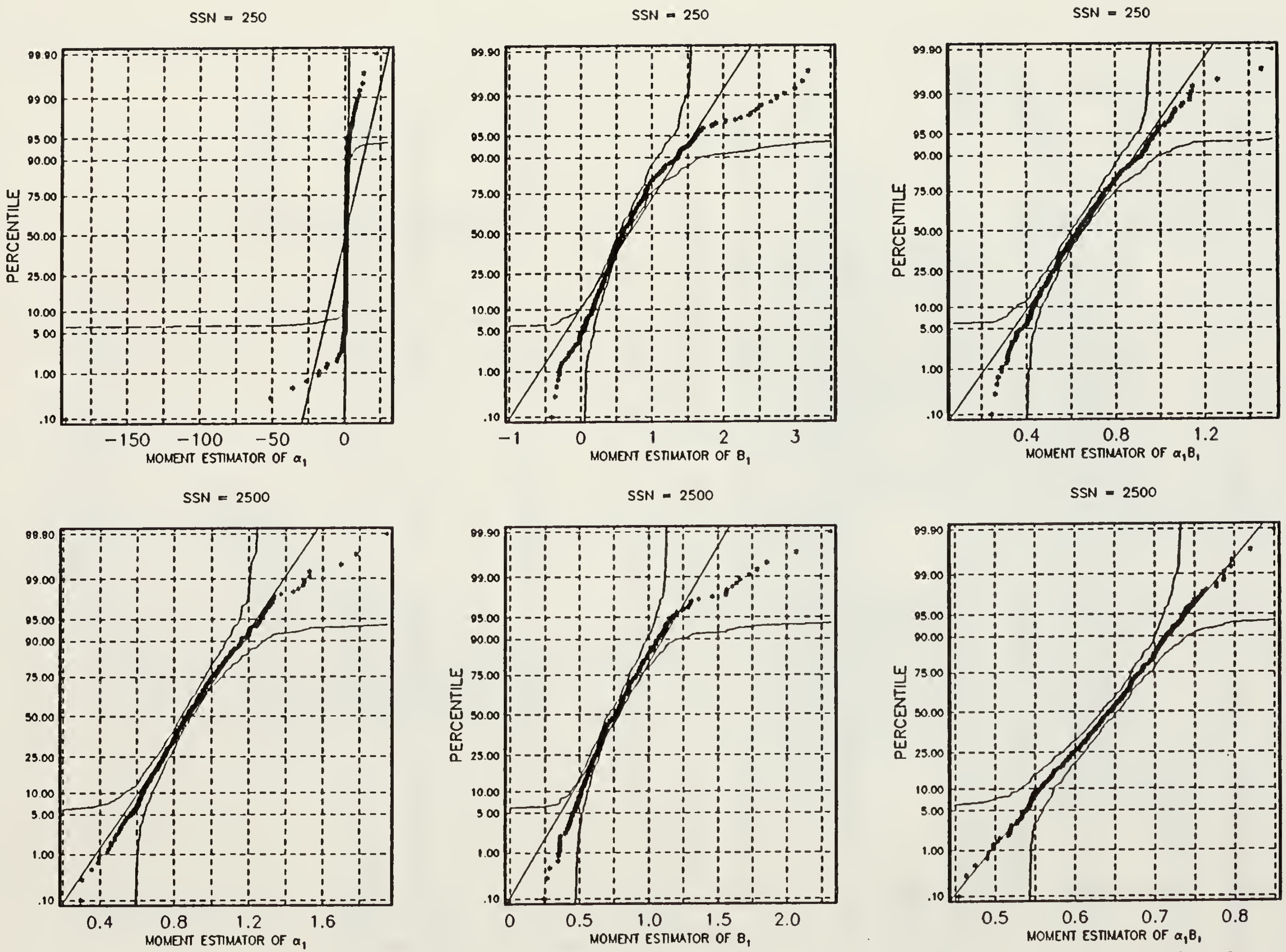


Figure II.D.4.3. Normal Probability Plots of Moment Estimators of α_1, β_1, and $\gamma = \alpha_1 \beta_1$ in the NLAR(1) Process for 500 Samples of Sizes 250 and 2500

However, a particularly useful way of looking at it is from

$$R_n = \beta_1(K'_n-\alpha_1)X_{n-1}+\varepsilon_n. \qquad \text{(II.D.4.11)}$$

It is from (II.D.4.11) that we see explicitly how the i.i.d. innovation, $\{\varepsilon_n\}$, and the coefficient $\{K'_n-\alpha_1\}$ processes impact on the linear residual.

Let $\mathcal{F}_{n-1}$ be the σ-algebra generated by $[\{(K'_k-\alpha_1),\varepsilon_k\};\ k=1,\ldots,n-1]$. Intuitively, $\mathcal{F}_{n-1}$, represents all the information about the process up to time n-1. Conditioning on $\mathcal{F}_{n-1}$, we have the following two useful properties of R_n as noted by Nicholls and Quinn [Ref. 16: p. 42].

$$E(R_n|\ \mathcal{F}_{n-1}) = 0. \qquad \text{(II.D.4.12)}$$

$$E(R_n^2|\ \mathcal{F}_{n-1}) = \beta_1^2\mathrm{Var}(K'_n)x_{n-1}^2 + \mathrm{Var}(\varepsilon_n) \qquad \text{(II.D.4.13)}$$

$$= \alpha_1(1-\alpha_1)\beta_1^2x_{n-1}^2 + 2(1-\alpha_1\beta_1^2) \qquad \text{(II.D.4.14)}$$

These results follow because X_{n-1} is a function only of the process through (n-1) and $(K'_n-\alpha_1)$ and ε_n are both independent of it.

(2) <u>The Least Squares Estimator of $\gamma = \alpha_1\beta_1$.</u> Using R_n from (II.D.4.10) and a given sample from $\{X_n\}$, we obtain the least squares estimator by minimizing the sum $\sum_{i=2}^{n} R_i^2$ with respect to the product $\alpha_1\beta_1$ which is now called γ. We have

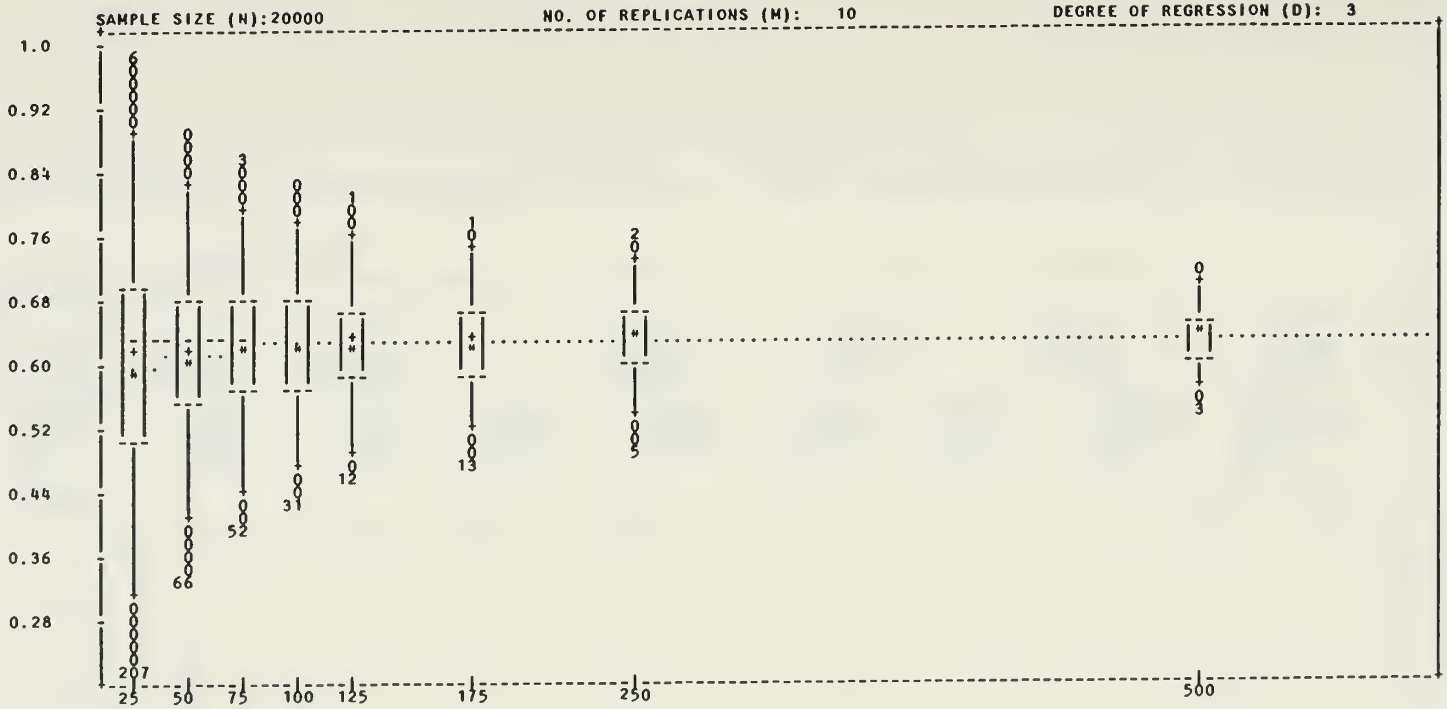
$$\hat{\gamma} = \frac{\sum_{i=2}^{n} X_i X_{i-1}}{\sum_{i=2}^{n} X_{i-1}^2} , \qquad \text{(II.D.4.15)}$$

which, in fact, is the usual expression for the estimation of serial correlation in linear AR(1) models as given, for example, in Chatfield [Ref. 31: p. 66].

Since the NLAR(1) process is an RCA(1) process of Nicholls and Quinn, it follows from their theorem [Ref. 16: p. 44] that $\hat{\gamma}$ is strongly consistent, asymptotically unbiased and $\frac{1}{\sqrt{N}}(\hat{\gamma}-\gamma)$ has an asymptotic Normal distribution. The asymptotic variance, from the same results of Nicholls and Quinn, is

$$\sigma^2_{\hat{\gamma}} = 1 + 5\alpha_1\beta_1^2 - 6(\alpha_1\beta_1)^2. \qquad \text{(II.D.4.16)}$$

Figures II.D.4.4-II.D.4.7 contain the boxplot analysis of SIMTBED [Ref. 15] output for selected choices of α_1 and β_1 in the simulation of the least squares estimator of the product $\alpha_1\beta_1$ in the NLAR(1) processes. Note that although the estimated asymptotic mean is the true value, $\gamma = \alpha_1\beta_1 = .64$, for each of the four sets of the parameters, the estimated asymptotic variance of the estimator of $\alpha_1\beta_1 = \gamma$ is different for each of the four different sets of parameters. The simulation results reflect the asymptotic theoretical results for the NLAR(1) processes as given above.

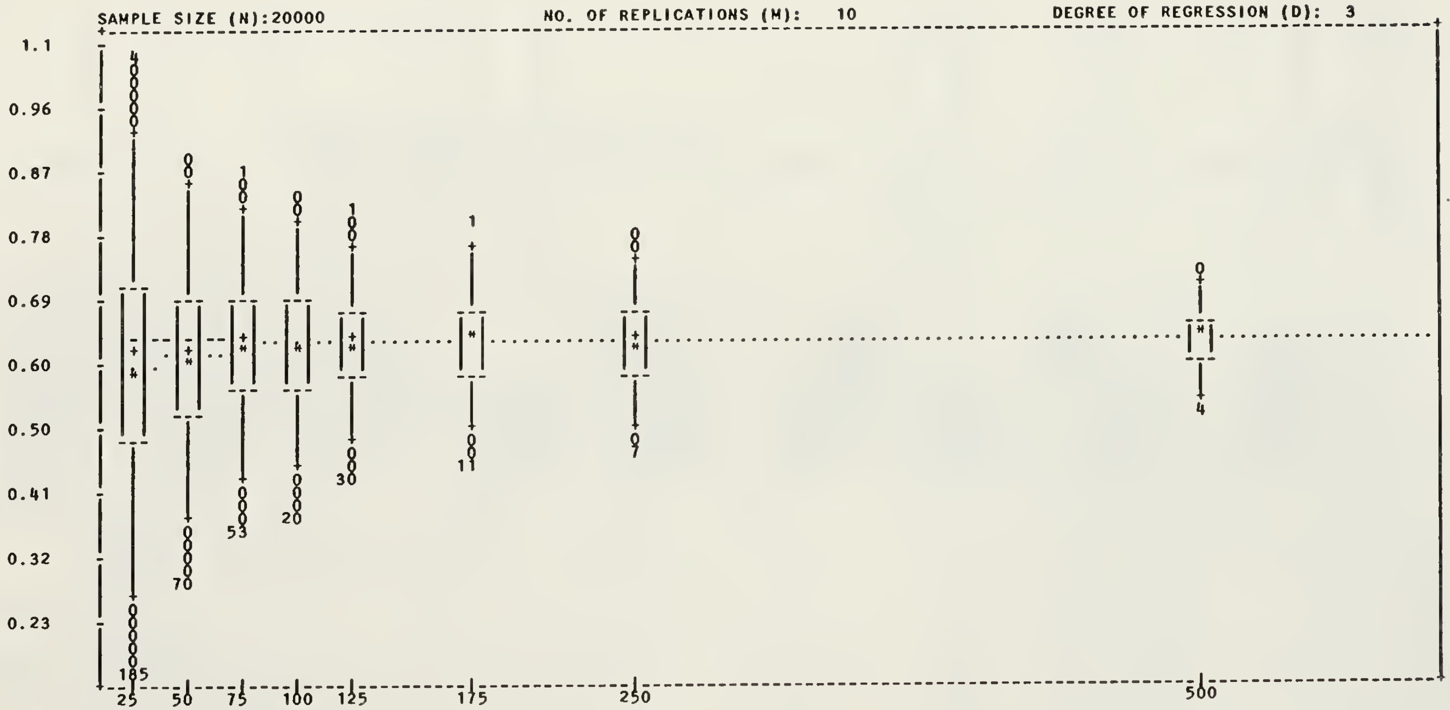


SUBSAMPLE SIZE	25	50	75	100	125	175	250	500
MEAN	0.5972	0.6179	0.6238	0.6278	0.6333	0.6336	0.6352	0.6358
STD	0.1584	0.1078	0.9150E-01	0.7759E-01	0.6659E-01	0.5690E-01	0.4915E-01	0.3316E-01
STD MEAN	0.1771E-02	0.1705E-02	0.1774E-02	0.1735E-02	0.1665E-02	0.1685E-02	0.1738E-02	0.1658E-02
SKEWNESS	-0.8774	-0.6975	-0.6688	-0.6216	-0.4584	-0.3387	-0.3174	-0.2920
KURTOSIS	1.2110	0.5706	0.8486	1.3381	0.3872	0.3105	0.1477	0.0628

MEAN OF REGRESSION ON AVERAGES - COEFFICIENTS:	0.637984	-0.639307	-30.3316	526.531
VARIANCE OF REGRESSION - COEFFICIENTS:	0.758701E-05	0.586319	2916.19	835611.
STD DEV OF REGRESSION - COEFFICIENTS:	0.275445E-02	0.765715	54.0018	914.117
REGRESSION ON VARIANCE - COEFFICIENTS:	0.393183	4.96948	-39.9838	104.874

ESTIMATOR: LEAST SQUARES USING LINEAR RESIDUAL FROM LAR(1) LINEAR PROCESS. RHO = 0.64000.
VERTICAL SCALE: YMIN = 0.2195
YMAX = 1.0004

Figure II.D.4.4. SIMTBED Boxplot Analysis of Least Squares Estimator of $\gamma=\alpha_1\beta_1$ in the LAR(1) Process with $\gamma=.64$

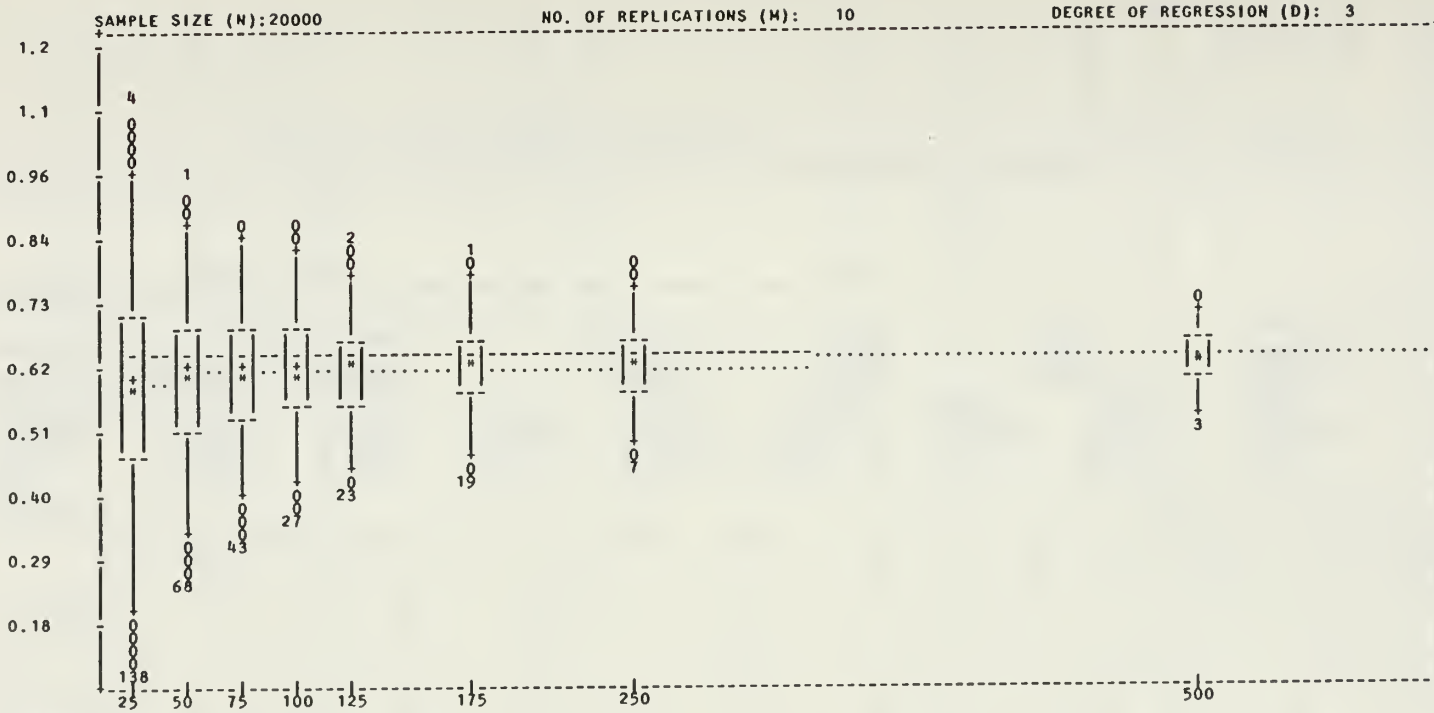


SUBSAMPLE SIZE	25	50	75	100	125	175	250	500
MEAN	0.5905	0.6145	0.6228	0.6262	0.6297	0.6348	0.6306	0.6374
STD	0.1784	0.1263	0.1006	0.8834E-01	0.7842E-01	0.6605E-01	0.5798E-01	0.4176E-01
STD MEAN	0.1995E-02	0.1997E-02	0.1951E-02	0.1975E-02	0.1961E-02	0.1956E-02	0.2050E-02	0.2088E-02
SKEWNESS	-0.8414	-0.7674	-0.6962	-0.5155	-0.5394	-0.4171	-0.3712	-0.5471
KURTOSIS	1.1265	0.8472	0.8179	0.3186	0.6113	0.3647	-0.0175	0.1879

MEAN OF REGRESSION ON AVERAGES - COEFFICIENTS:	0.638529	-1.06769	-10.0285	171.747
VARIANCE OF REGRESSION - COEFFICIENTS:	0.565847E-05	0.310574	1815.23	594410.
STD DEV OF REGRESSION - COEFFICIENTS:	0.237875E-02	0.557292	42.6056	770.980
REGRESSION ON VARIANCE - COEFFICIENTS:	1.26750	-12.4173	97.6750	-236.882

ESTIMATOR: LEAST SQUARES USING LINEAR RESIDUAL FROM NLAR(1) PROCESS; RHO = 0.64; ALPHA = .9; BETA = 0.7111.
VERTICAL SCALE: YMIN = 0.1551
YMAX = 1.0568

Figure II.D.4.5. SIMTBED Boxplot Analysis of Least Squares Estimator of $\gamma=\alpha_1\beta_1$ with $\alpha_1= .9$ and $\beta_1=.7\overline{1}$ in the NLAR(1) Process

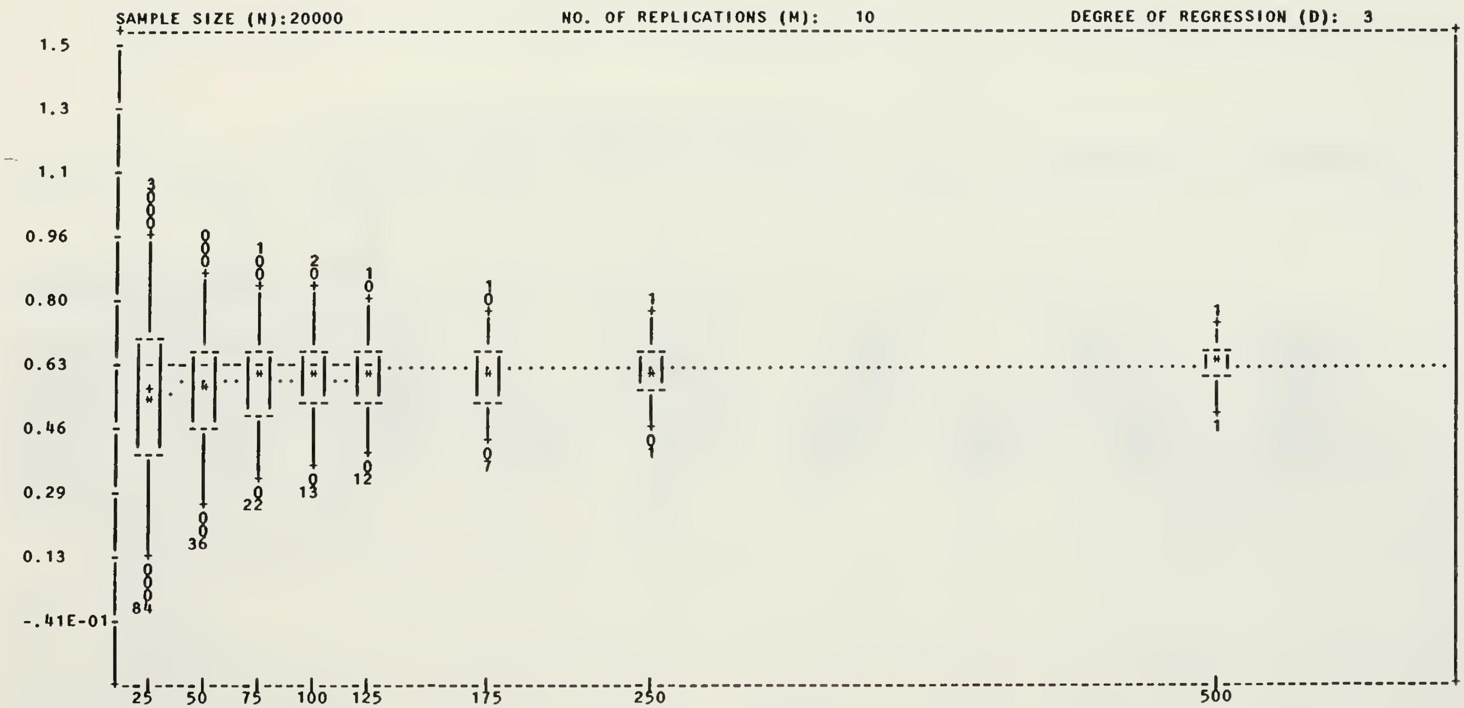


SUBSAMPLE SIZE	25	50	75	100	125	175	250	500
MEAN	0.5812	0.6073	0.6173	0.6213	0.6240	0.6280	0.6310	0.6358
STD	0.1945	0.1376	0.1135	0.1009	0.8899E-01	0.7988E-01	0.6882E-01	0.4911E-01
STD MEAN	0.2174E-02	0.2176E-02	0.2201E-02	0.2255E-02	0.2225E-02	0.2366E-02	0.2433E-02	0.2456E-02
SKEWNESS	-0.7158	-0.6428	-0.6184	-0.5450	-0.5017	-0.5194	-0.3614	-0.4451
KURTOSIS	0.8006	0.3854	0.4887	0.2539	0.4279	0.6823	0.1063	0.0112

MEAN OF REGRESSION ON AVERAGES - COEFFICIENTS:	0.639411	-2.12999	37.5880	-513.490
VARIANCE OF REGRESSION - COEFFICIENTS:	0.536177E-05	0.307858	1168.45	294714.
STD DEV OF REGRESSION - COEFFICIENTS:	0.231555E-02	0.554849	34.1826	542.876
REGRESSION ON VARIANCE - COEFFICIENTS:	1.65682	-11.4888	59.7070	-100.185

ESTIMATOR: LEAST SQUARES USING LINEAR RESIDUAL FROM NLAR(1) PROCESS; RHO = 0.64; ALPHA = .8; BETA = 0.8.
VERTICAL SCALE: YMIN = 0.0876
YMAX = 1.1789

Figure II.D.4.6. SIMTBED Boxplot Analysis of Least Squares Estimator of $\gamma=\alpha_1\beta_1$ with $\alpha_1=\beta_1=.8$ in the NLAR(1) Process



SUBSAMPLE SIZE	25	50	75	100	125	175	250	500
MEAN	0.5501	0.5849	0.6007	0.6068	0.6095	0.6166	0.6200	0.6342
STD	0.2065	0.1477	0.1227	0.1122	0.1007	0.8879E-01	0.7480E-01	0.5406E-01
STD MEAN	0.2309E-02	0.2335E-02	0.2378E-02	0.2508E-02	0.2518E-02	0.2630E-02	0.2645E-02	0.2703E-02
SKEWNESS	-0.5243	-0.4591	-0.3025	-0.2862	-0.3734	-0.2100	0.0500	-0.0545
KURTOSIS	0.2795	0.2228	0.0678	-0.0946	0.5951	0.0466	-0.1798	0.0382

MEAN OF REGRESSION ON AVERAGES - COEFFICIENTS:	0.639751	-4.54010	130.122	-1814.28
VARIANCE OF REGRESSION - COEFFICIENTS:	0.166045E-04	0.565485	2060.92	515714.
STD DEV OF REGRESSION - COEFFICIENTS:	0.407486E-02	0.751988	45.3974	718.133
REGRESSION ON VARIANCE - COEFFICIENTS:	1.62826	-1.77706	-38.5250	166.904

ESTIMATOR: LEAST SQUARES USING LINEAR RESIDUAL FROM TLAR(1) PROCESS; RHO = 0.6400.
VERTICAL SCALE: YMIN = -0.1745
YMAX = 1.4647

Figure II.D.4.7. SIMTBED Boxplot Analysis of Least Squares Estimator of $\gamma=\alpha_1\beta_1$ with $\gamma=.64$ in the TLAR(1) Process

An analogous result is given in Section III.E.4, where the theory of least squares is derived for the Beta-Laplace AR(1) model.

(3) <u>The Joint Least Squares Estimation for α_1 and β_1.</u> It is not possible to minimize $\sum_{i=2}^{n} R_i^2$ with respect to α_1 and β_1 individually. However, a technique from Nicholls and Quinn [Ref. 16: p. 43], which uses the result in (II.D.4.13) is applicable. As was pointed out earlier in Section II.C.2, by assuming nothing about the particular marginal distribution, Nicholls and Quinn were free to treat the variances, σ_ε^2 and $\sigma_{K'}^2$, as completely independent parameters subject only to the constraint that the marginal distribution of $\{X_n\}$, whatever it is, has a positive variance. Then, given (II.D.4.13), it was possible to estimate σ_ε^2 and $\sigma_{K'}^2$ by minimizing the sum of squares $\sum_{i=2}^{n} \overline{S}_i^2$ where

$$\overline{S}_n = \hat{R}_n^2 - \sigma_\varepsilon^2 - \sigma_{X'}^2 X_{n-1}^2, \qquad \text{(II.D.4.17)}$$

and $\hat{R}_n^2 = (X_n - \hat{\gamma} X_{n-1})^2$ and $\hat{\gamma}$ is from (II.D.4.15). They derive the properties of the trivariate distribution of the estimator of $(\gamma, \sigma_\varepsilon^2, \sigma_{K'}^2)$.

Since σ_ε^2 and $\sigma_{K'}^2$ are related parametrically in α_1 and β_1, the results in [Ref. 16] concerning the variances do not apply in the NLAR(1) process. However, we can form from (II.D.4.13) and (II.D.4.10) an analogous expression for

$$S_n = R_n^2 - \alpha_1\beta_1^2(1-\alpha_1)X_{n-1}^2 - 2(1-\alpha_1\beta_1^2), \qquad \text{(II.D.4.18)}$$

where the product $\alpha_1\beta_1$ in R_n is not replaced by $\hat{\gamma}$ from (II.D.4.15).

In terms of a sample from $\{X_n\}$, we define the joint least squares estimators of α_1 and β_1 to be those values $\hat{\alpha}_1$ and $\hat{\beta}_1$ that minimize

$$\sum_{i=2}^{n}\{(x_i-\alpha_1\beta_1x_{i-1})^2 - \alpha_1(1-\alpha_1)\beta_1^2x_{i-1}^2 - 2(1-\alpha_1\beta_1)\}^2, \qquad \text{(II.D.4.19)}$$

where (II.D.4.19) is the sum of the squares of S_n given in (II.D.4.18). Now it is clear that (II.D.4.19) is a highly nonlinear expression in two unknowns, α_1 and β_1. A given numerical technique could converge to a local extremum, a saddle point, or diverge depending on, among other things, the starting values for estimating α_1 and β_1.

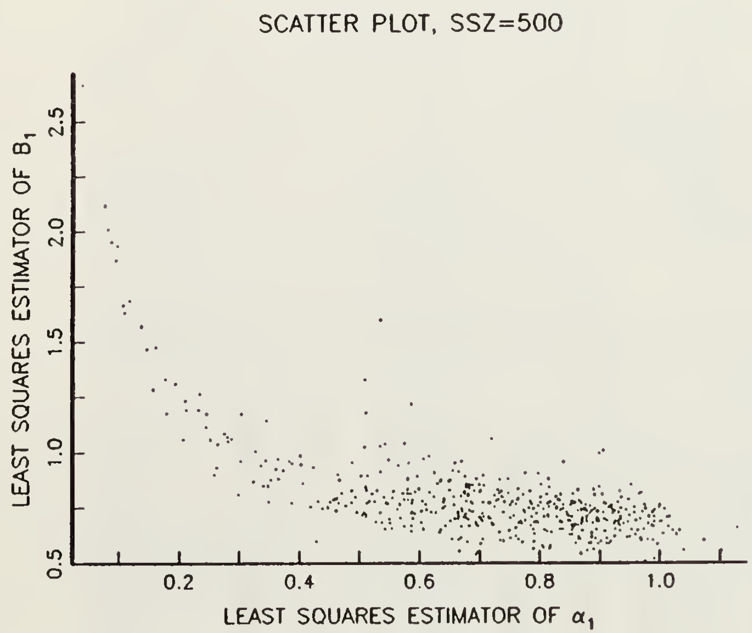
Constraining the nonlinear optimization problem given by (II.D.4.19) to the rectangle within which the NLAR(1) process is defined--$0 \leq \alpha_1 \leq 1$ and $-1 \leq \beta_1 \leq 1$--eliminates the divergence problem, but clouds the estimation issue regarding the boundary models LAR(1) and TLAR(1). We try an unconstrained approach described below.

(4) <u>An Unconstrained Nonlinear Optimization of (II.D.4.19)</u>. It is easy, but tedious, to write the normal equations from (II.D.4.19). One critical point is at $\alpha_1 = \beta_1 = 0$. After factoring α_1 from the one equation and β_1 from the second, several iterations of the Newton-Raphson method (see, for example, Gerald [Ref. 28: pp. 122-128]) can be performed to find other critical points. The Newton-Raphson method uses

a second-order Taylor series approximation to solve the non-linear system by a set of linear Jacobian equations. However, one needs to calculate the four second partial derivatives from (II.D.4.19) and to have a good starting point on the surface.

The IMSL routine ZSPOW solves systems of non-linear equations for one root using modified Newton methods. This routine was used to solve the unconstrained problem of finding $\hat{\alpha}_1$ and $\hat{\beta}_1$ from sets of data from simulated NLAR(1) processes. The routine was very senstive to starting values and did not always converge even when the sample size was as large as 2500. It also did not perform well when the true correlation coefficient, $\gamma = \alpha_1\beta_1$, was small for any of the simulated NLAR(1) processes with the same autocorrelation function, $\gamma^{|k|}$. This problem is highlighted by the fact that (II.D.4.19) is constant along the line $\alpha_1 = 0$ and the line $\beta_1 = 0$.

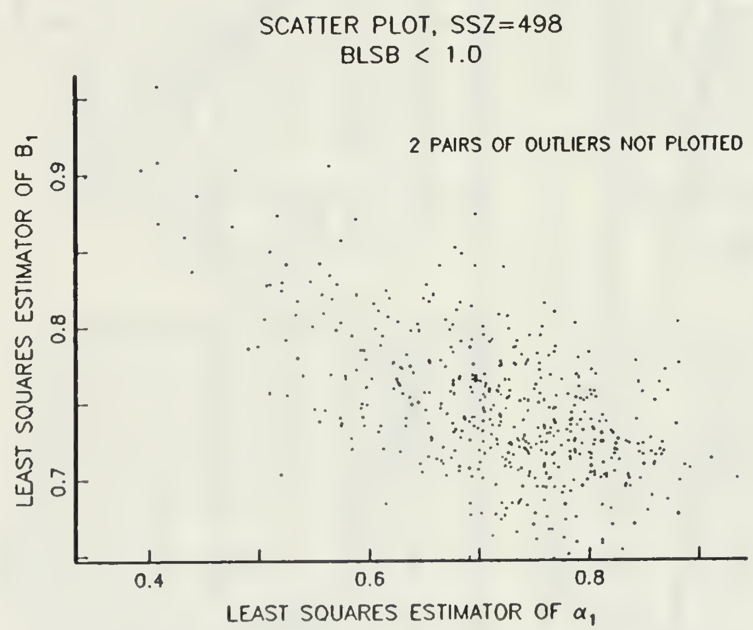
As an illustration of the performance of the routine, 500 sets of sample sizes 250 and 2500, respectively, were generated from the NLAR(1) process with $\alpha_1 = \beta_1 = .8$. The scatter plot analyses in Figures II.D.4.8 and II.D.4.9 show how the estimators $\hat{\alpha}_1$ and $\hat{\beta}_1$ determined by ZSPOW are related. Especially for the samples of size 250, there is the same pattern of the hyperbola as seen in the moment estimators of α_1 and β_1 given in Section II.D.4.b.(2). From the accompanying tables, it is clear that the variance of the marginal distributions for each estimator $\hat{\alpha}_1$ and $\hat{\beta}_1$ is decreasing with increased sample size. The Normal plots of the empirical marginal cumulative distribution functions for $\hat{\alpha}_1$ and for $\hat{\beta}_1$ appear very non-Normal even



SCATTER PLOT TABLE

X	:ALSA
Y	:BLSA
SELECTION	:ALL
X LABEL	:LEAST SQUARES ESTIMATOR OF α_1
Y LABEL	:LEAST SQUARES ESTIMATOR OF B_1
NO. OF ELEMENTS	:500
CORRELATION XY	:⁻0.68532
RK CORRELATION	:⁻0.56881 T=⁻15.433
X MEAN	:0.70755
STD. DEVIATION	:0.21506
5-PERCENTILE	:0.25842
25-PERCENTILE	:0.58544
MEDIAN	:0.73755
75-PERCENTILE	:0.87119
95-PERCENTILE	:0.9858
X MIN.	:0.0388 0.07631 0.08116
X MAX.	:1.1283 1.1022 1.073
Y MEAN	:0.81115
STD. DEVIATION	:0.21997
5-PERCENTILE	:0.61106
25-PERCENTILE	:0.70478
MEDIAN	:0.76518
75-PERCENTILE	:0.84429
95-PERCENTILE	:1.1949
Y MIN	:0.53098 0.54016 0.56256
Y MAX	:2.687 2.1393 2.0339

Figure II.D.4.8. Scatter Plot Analysis of Joint Least Squares Estimators of (α_1, β_1) in the NLAR(1) Process for 500 Samples of Size 250 with $\alpha_1=\beta_1=.8$



SCATTER PLOT TABLE

X	:ALSB
Y	:BLSB
SELECTION	:BLSB < 1.0
X LABEL	:LEAST SQUARES ESTIMATOR OF α_1
Y LABEL	:LEAST SQUARES ESTIMATOR OF B_1
NO. OF ELEMENTS	:498
CORRELATION XY	:⁻0.60726
RK CORRELATION	:⁻0.52896 T=⁻13.881
X MEAN	:0.71526
STD. DEVIATION	:0.095181
5-PERCENTILE	:0.53311
25-PERCENTILE	:0.66911
MEDIAN	:0.72617
75-PERCENTILE	:0.78287
95-PERCENTILE	:0.85297
X MIN.	:0.34202 0.39302 0.40738
X MAX.	:0.9348 0.91104 0.89164
Y MEAN	:0.74974
STD. DEVIATION	:0.047153
5-PERCENTILE	:0.68047
25-PERCENTILE	:0.71897
MEDIAN	:0.74204
75-PERCENTILE	:0.77426
95-PERCENTILE	:0.83713
Y MIN	:0.65086 0.65195 0.65365
Y MAX	:0.96088 0.91134 0.90841

Figure II.D.4.9. Scatter Plot Analysis of Joint Least Squares Estimators of (α_1,β_1) in the NLAR(1) Process for 500 Samples of Size 2500 with $\alpha_1=\beta_1=.8$

from estimators derived from samples of 2500. On the other hand, the Normal plots of $\hat{\gamma} = \hat{\alpha}_1 \hat{\beta}_1$ indicate that the distribution is converging to a Normal distribution as required by theoretical results of the previous subsection. (See Figure II.D.4.10).

It is convenient, at this point, to summarize the results on the moment and least squares estimation of $\gamma = \alpha_1 \beta_1$ and (α_1, β_1) in the NLAR(1) processes.

In the estimation of γ, only second-order product moments are required for both methods. From the Normal probability plots in Figures II.D.4.3 and II.D.4.10, it appears that both estimators of γ are converging to Normal distributions. Although the moment estimator of γ is unbiased (the least squares estimator is asymptotically unbiased), the variance of the moment estimator of γ is considerably larger than that of the least squares estimator of γ.

The estimation of α_1 and β_1 requires fourth-order product moments for both methods. The variance of the moment estimators of α_1 and β_1 are too large, even for samples of size 2500 to be useful in distinguishing between NLAR(1) processes. The least squares estimators of α_1 and β_1 have smaller variances than the corresponding moment estimators and could be useful in distinguishing between NLAR(1) processes. However, as pointed out above, the numerical routine to find the critical points does not always converge for a given starting value of α_1 and β_1. The conclusion is that neither method of estimating α_1 and β_1 is very satisfactory.

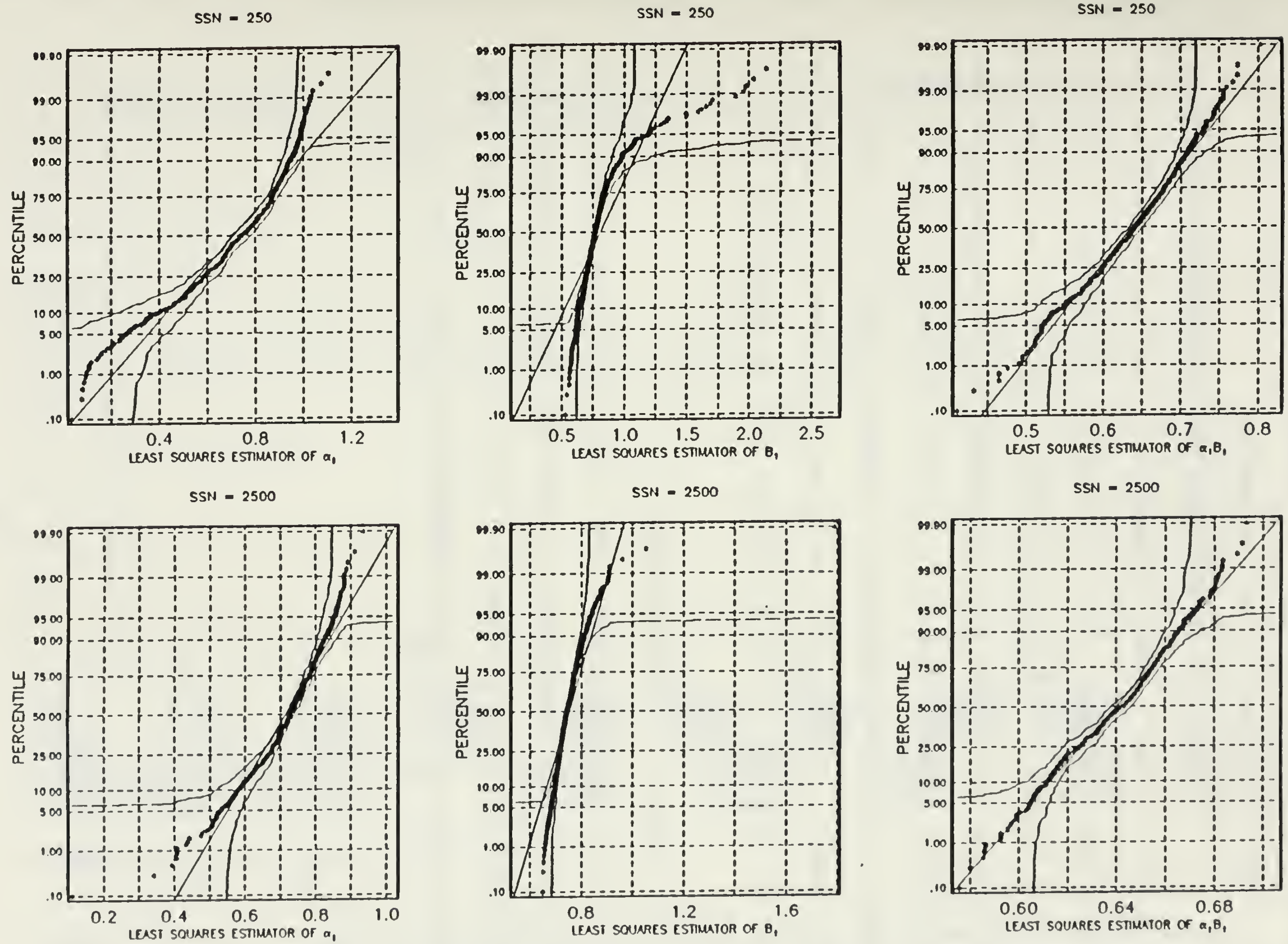


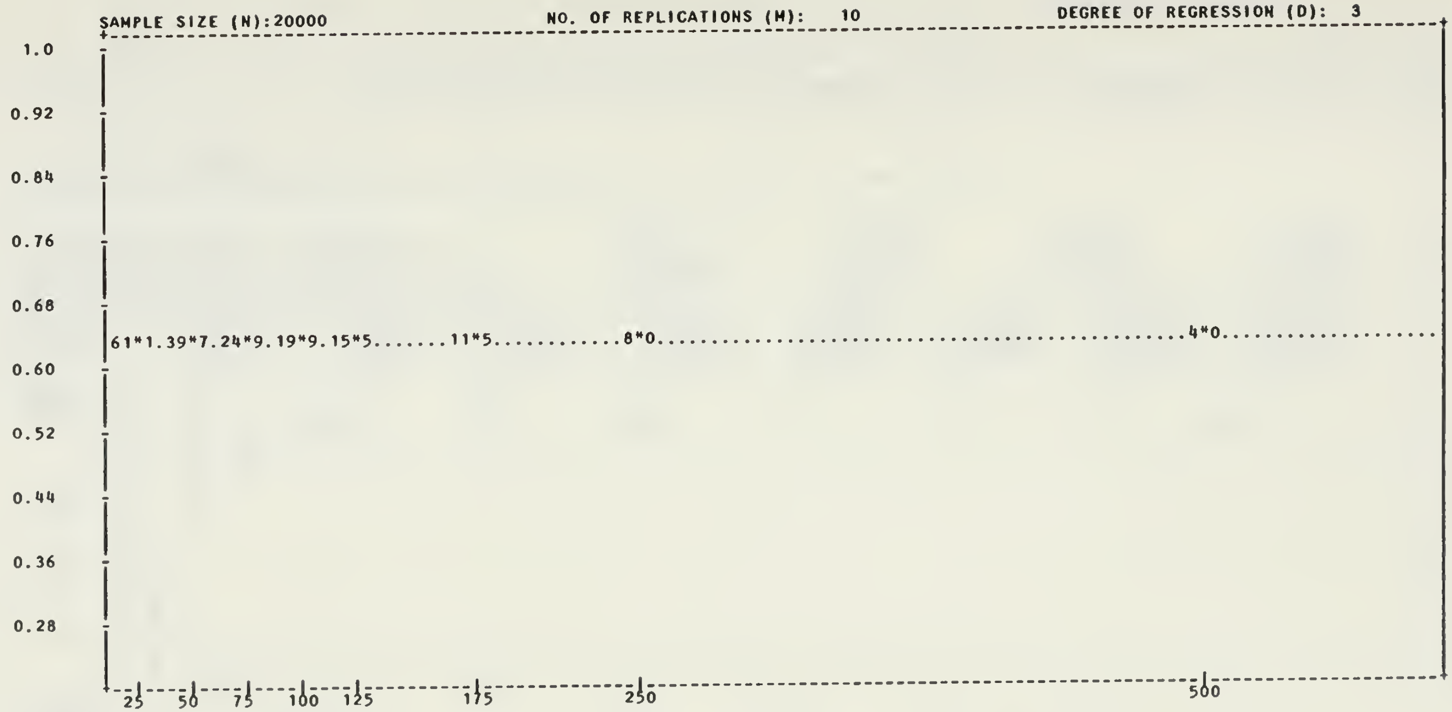
Figure II.D.4.10. Normal Probability Plots of the Least Squares Estimators of α_1, β_1 and $\gamma=\alpha_1\beta_1$ in the NLAR(1) Process for 500 Samples of Sizes 250 and 2500 with $\alpha_1=\beta_1=.8$

(5) <u>The Median (X_i/X_{i-1}) Estimator of $\gamma = \alpha_1\beta_1$.</u> The median of (X_i/X_{i-1}) was seen to be extremely efficient in the LAR(1) process. It also makes sense in the context of maximum likelihood estimation in LAR(1). This is discussed in the next section.

Simulation results confirm the conjecture that the median (X_i/X_{i-1}) is not a robust estimator of γ for departures from the LAR(1) process. In fact, from the boxplots in Figures II.D.4.11 - II.D.4.14 of SIMTBED output for four NLAR(1) processes, the estimators seem to become more biased as β_1 approaches one--corresponding to the other boundary process, TLAR(1). Even for the small size of the simulations, the standard deviation of the mean is small. For the three non-LAR(1) models, the asymptotic estimates of the mean of γ given in the data are each significantly different from the theoretical value of $\gamma = .64$.

d. Method of Maximum Likelihood

(1) <u>Introduction</u>. The logarithm of the likelihood function, $L(\alpha_1, \beta_1)$, is obtained by taking the natural logarithm of the n-dimensional joint density given in (II.D.2.4) and treating it as a function of α_1 and β_1 for a given realization of length n from $\{X_n\}$. We

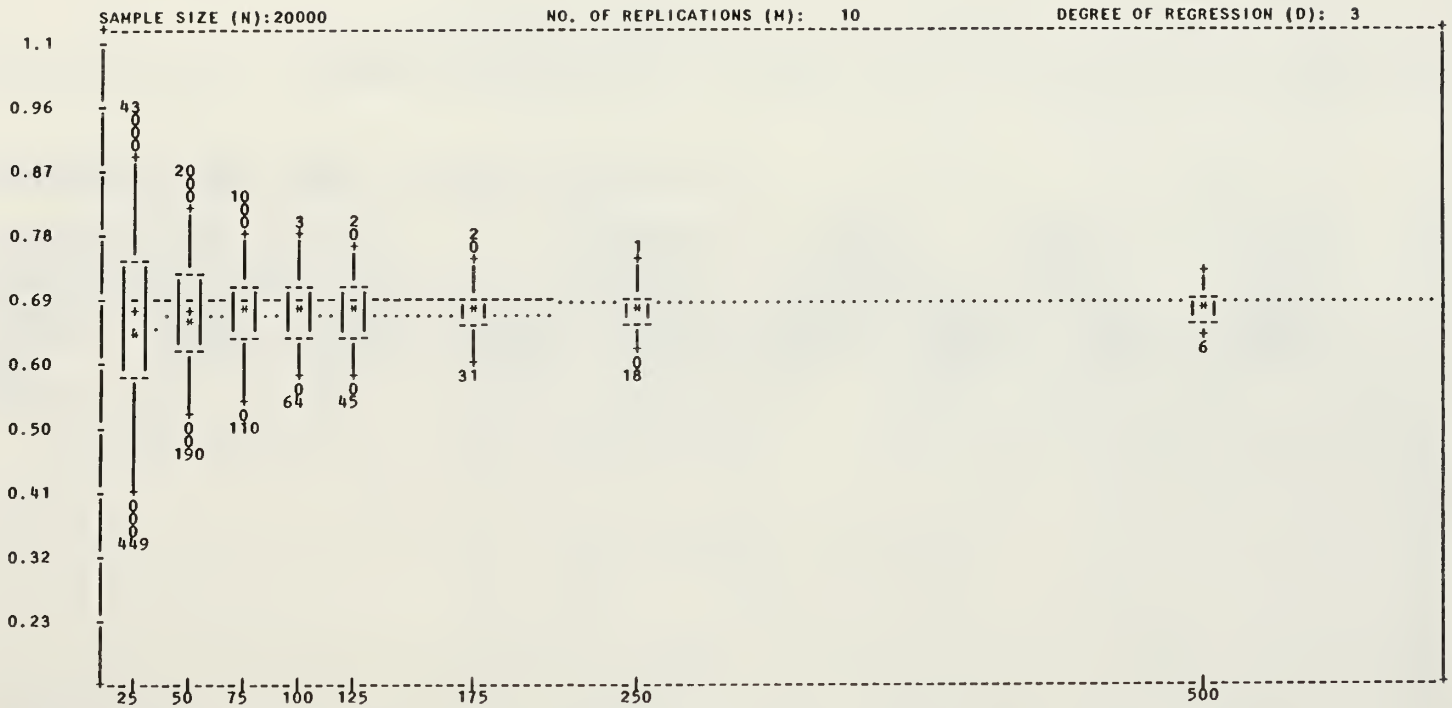


SUBSAMPLE SIZE	25	50	75	100	125	175	250	500
MEAN	0.6380	0.6398	0.6399	0.6399	0.6399	0.6399	0.6399	0.6400
STD	0.3113E-01	0.4389E-02	0.9123E-04	0.8600E-04	0.8060E-04	0.6988E-04	0.5395E-04	0.5166E-06
STD MEAN	0.3481E-03	0.6940E-04	0.1769E-05	0.1923E-05	0.2015E-05	0.2070E-05	0.1908E-05	0.2583E-07
SKEWNESS	-6.2711	-35.7408	1.0006	1.0008	1.0009	1.0013	1.0019	1.0083
KURTOSIS	199.1767	1632.3586	-2.0015	-2.0020	-2.0025	-2.0035	-2.0050	-1.9982

MEAN OF REGRESSION ON AVERAGES - COEFFICIENTS:	0.639997	-0.852690E-03	0.441628	-31.2276
VARIANCE OF REGRESSION - COEFFICIENTS:	0.211928E-07	0.115371E-03	1.00481	358.692
STD DEV OF REGRESSION - COEFFICIENTS:	0.145578E-03	0.107411E-01	1.00240	18.9392
REGRESSION ON VARIANCE - COEFFICIENTS:	-0.142404E-01	0.526420	-6.04652	21.8794

ESTIMATOR: MEDIAN X(I)/X(I-1) FROM LAR(1) PROCESS. RHO = 0.64000.
VERTICAL SCALE: YMIN = 0.2195
YMAX = 1.0004

Figure II.D.4.11. SIMTBED Boxplot Analysis of Median (X_i/X_{i-1}) Estimator of $\gamma=\alpha_1\beta_1$ with $\gamma=.64$ in the LAR(1) Process

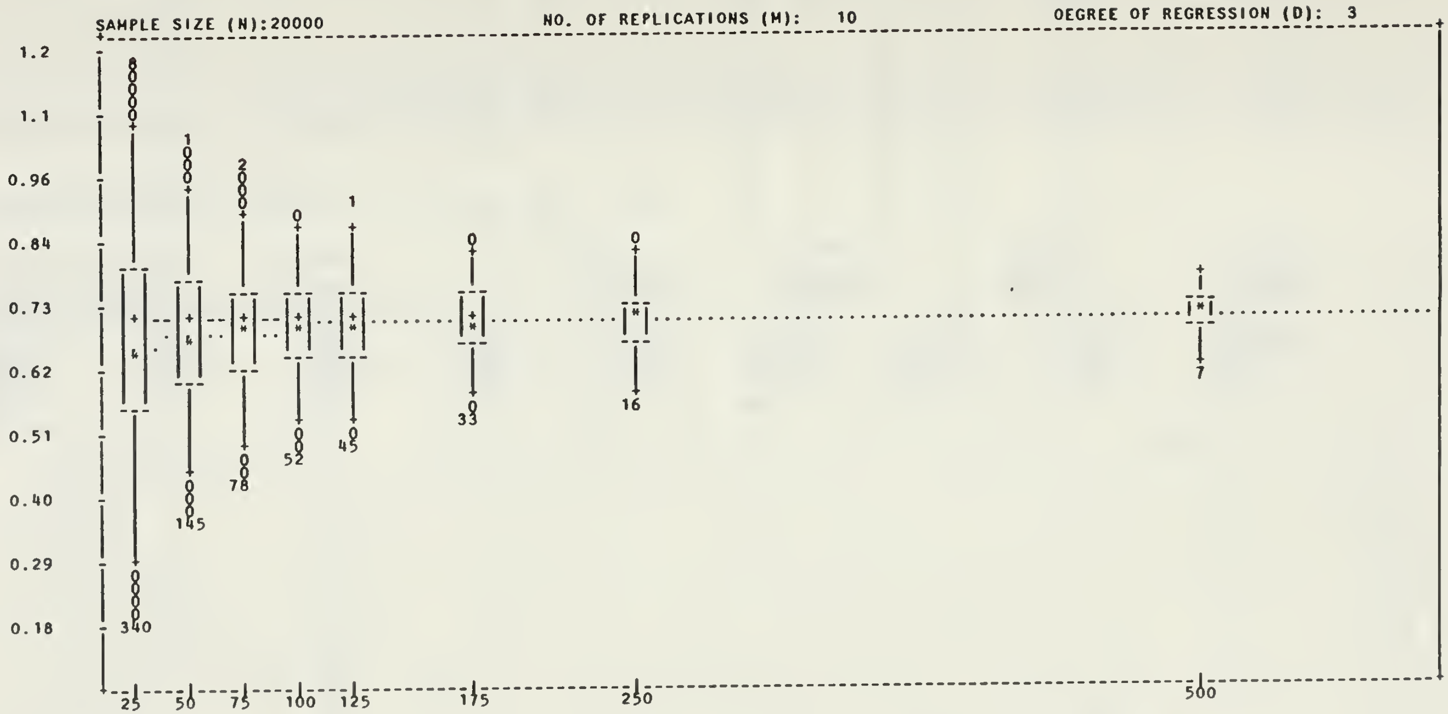


SUBSAMPLE SIZE	25	50	75	100	125	175	250	500
MEAN	0.6476	0.6670	0.6711	0.6767	0.6763	0.6785	0.6799	0.6820
STD	0.1615	0.9981E-01	0.7485E-01	0.6048E-01	0.5263E-01	0.4356E-01	0.3758E-01	0.2559E-01
STD MEAN	0.1805E-02	0.1578E-02	0.1451E-02	0.1352E-02	0.1316E-02	0.1290E-02	0.1329E-02	0.1279E-02
SKEWNESS	-1.3886	-1.3405	-1.2929	-1.2190	-0.9984	-0.9250	-1.0004	-0.5395
KURTOSIS	3.5447	3.7817	3.8965	3.5406	2.1277	1.8601	2.5560	0.8396

MEAN OF REGRESSION ON AVERAGES - COEFFICIENTS:	0.683592	-0.868505	3.76817	-106.861
VARIANCE OF REGRESSION - COEFFICIENTS:	0.179079E-05	0.110610	461.445	142863.
STD DEV OF REGRESSION - COEFFICIENTS:	0.133820E-02	0.332581	21.4812	377.972
REGRESSION ON VARIANCE - COEFFICIENTS:	0.564575	-8.10081	81.1578	-192.374

ESTIMATOR: MEDIAN X(I)/X(I-1) FROM NLAR(1) PROCESS; RHO = 0.64; ALPHA = 0.9; BETA = 0.7111.
VERTICAL SCALE: YMIN = 0.1551
YMAX = 1.0568

Figure II.D.4.12. SIMTBED Boxplot Analysis of Median (X_i/X_{i-1}) Estimator of $\gamma=\alpha_1\beta_1$ with $\alpha_1=.9$ and $\beta_1=.7\overline{1}$ in the NLAR(1) Process

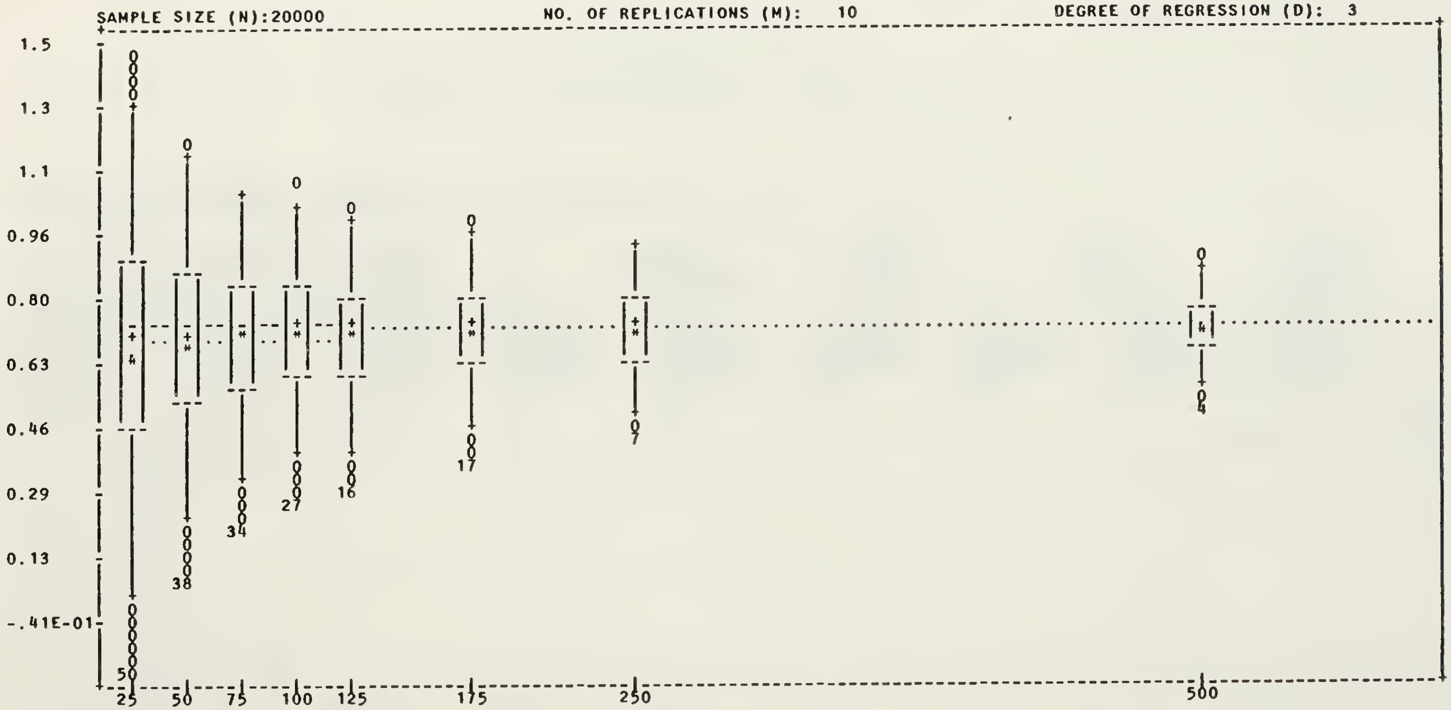


SUBSAMPLE SIZE	25	50	75	100	125	175	250	500
MEAN	0.6602	0.6869	0.6932	0.7021	0.7031	0.7065	0.7116	0.7133
STD	0.2170	0.1406	0.1156	0.9255E-01	0.8455E-01	0.7125E-01	0.5919E-01	0.3867E-01
STD MEAN	0.2426E-02	0.2223E-02	0.2242E-02	0.2069E-02	0.2114E-02	0.2110E-02	0.2093E-02	0.1933E-02
SKEWNESS	-1.0978	-1.1698	-1.2186	-1.1607	-1.0657	-0.8897	-0.7462	-0.6515
KURTOSIS	1.5582	1.8745	2.6435	2.3466	1.6030	1.1471	0.7048	0.6911

MEAN OF REGRESSION ON AVERAGES - COEFFICIENTS:	0.717981	-2.02791	30.0274	-378.330
VARIANCE OF REGRESSION - COEFFICIENTS:	0.986523E-05	0.542296	1952.32	484618.
STD DEV OF REGRESSION - COEFFICIENTS:	0.314089E-02	0.736407	44.1851	696.146
REGRESSION ON VARIANCE - COEFFICIENTS:	0.547685	6.88180	-44.1438	127.331

ESTIMATOR: MEDIAN X(I)/X(I-1) FROM NLAR(1) PROCESS; RHO = 0.64; ALPHA = 0.8; BETA = 0.8.
VERTICAL SCALE: YMIN = 0.0876
YMAX = 1.1789

Figure II.D.4.13. SIMTBED Boxplot Analysis of Median (X_i/X_{i-1}) Estimator of $\gamma=\alpha_1\beta_1$ with $\alpha_1=\beta_1=.8$ in the NLAR(1) Process



SUBSAMPLE SIZE	25	50	75	100	125	175	250	500
MEAN	0.6608	0.6930	0.6977	0.7127	0.7111	0.7184	0.7219	0.7224
STD	0.3041	0.2228	0.1883	0.1614	0.1494	0.1246	0.1056	0.7713E-01
STD MEAN	0.3400E-02	0.3522E-02	0.3651E-02	0.3610E-02	0.3735E-02	0.3692E-02	0.3732E-02	0.3857E-02
SKEWNESS	-0.6751	-0.6958	-0.7126	-0.6482	-0.5714	-0.5134	-0.4946	-0.2116
KURTOSIS	-0.1009	0.0884	0.2681	0.1246	0.1363	0.3374	0.1465	0.0215

MEAN OF REGRESSION ON AVERAGES - COEFFICIENTS:	0.728410	-2.04246	9.93394	-19.6258
VARIANCE OF REGRESSION - COEFFICIENTS:	0.174946E-04	0.895999	4960.70	0.151741E+07
STD DEV OF REGRESSION - COEFFICIENTS:	0.418266E-02	0.946572	70.4322	1231.83
REGRESSION ON VARIANCE - COEFFICIENTS:	3.05163	-2.70753	-16.9508	59.8585

ESTIMATOR: MEDIAN X(I)/X(I-1) FROM TLAR(1) PROCESS; RHO = 0.6400.
VERTICAL SCALE: YMIN = -0.1745
YMAX = 1.4647

Figure II.D.4.14. SIMTBED Boxplot Analysis of Median (X_i/X_{i-1}) Estimator of $\gamma=\alpha_1\beta_1$ with $\gamma=.64$ in the TLAR(1) Process

have

$$L(\alpha_1,\beta_1) = -n(\ell n2) - |x_1| + \sum_{i=2}^{n} \ell n[\alpha_1(1-p_2)\exp\{-|x_i-\beta_1 x_{i-1}|\}$$

$$+ (1-\alpha_1)(1-p_2)\exp\{-|x_i|\} + \alpha_1 p_2 \lambda \exp\{-\lambda|x_i-\beta_1 x_{i-1}|\}$$

$$+ (1-\alpha_1)p_2\lambda\exp\{-\lambda|x_i|\}], \qquad \text{(II.D.4.20)}$$

where p_2 was given in (II.D.13) and $\lambda = \frac{1}{\sqrt{(1-\alpha_1)\beta_1^2}}$.

Maximizing (II.D.4.20) in the general NLAR(1) model is not accomplished here for two reasons. First, $L(\alpha_1,\beta_1)$ is not differentiable with respect to β_1 at any of the n values $\beta_1 = x_i/x_{i-1}$ for $i = 1,\ldots,n$, because of the terms $|x_i-\beta_1 x_{i-1}|$. A bivariate search routine that does not use derivatives is needed.

Second, $L(\alpha_1,\beta_1)$ is not defined along the line $\alpha_1 = 1$ at any of $0 \leqq k \leqq n$ values of β_1 such that $-1 < \beta_1 = x_i/x_{i-1} < 1$. To see this, examine the third term of the natural logarithm in (II.D.4.20). We have replacing λ for all $i = 2,\ldots,n$

$$\frac{\alpha_1 p_2}{\sqrt{(1-\alpha_1)\beta_1^2}} \exp\left\{\frac{-|x_i-\beta_1 x_{i-1}|}{\sqrt{(1-\alpha_1)\beta_1^2}}\right\}. \qquad \text{(II.D.4.21)}$$

Because of the presence of the exponential term in (II.D.4.21), the limit as α_1 approaches one is zero, so long as $\beta_1 \neq x_i/x_{i-1}$. The limit does not exist on the set $B = \{\beta_1 | \beta_1 = x_i/x_{i-1};\ i = 2,\ldots,n\}$.

It is worth noting that for $\alpha_1 = 1$, corresponding to the LAR(1) model, and except on the set B, (II.D.4.20), can be written as

$$L(1,\beta_1) = -n(\ln 2) - |x_1| + (n-1)\ln(1-\beta_1^2)$$

$$- \sum_{i=2}^{n} |x_i - \beta_1 x_{i-1}|, \quad \beta_1 \notin B. \qquad \text{(II.D.4.22)}$$

Now $\ln(1-\beta_1^2)$ is maximized at $\beta_1 = 0$ and the optimal value for $\sum_{i=2}^{n} |x_i - \beta_1 x_{i-1}|$ is the least absolute deviation (LAD) estimator of β_1 which is the weighted median of (x_i/x_{i-1}) where the weights are x_{i-1} for $i = 2,\ldots,n$. Thus, if after a large number of observations from $\{X_n\}$ no repeats of x_i/x_{i-1} are observed, then there will be little difference between a particular LAR(1) model and the completely random model of i.i.d. Laplace variables. In this case, for any β_1 in a small <u>deleted</u> neighborhood around $\hat{\beta}_1 = \text{med}(x_i/x_{i-1})$, (II.D.4.22) will be large because both $\ln(1-\beta_1^2)$ and $\sum_{i=1}^{n} |x_i - \beta_1 x_{i-1}|$ will be optimized.

(2) <u>The Maximum Likelihood Estimator of α_1 in the TLAR(1) Processes</u>. In this section, the likelihood function for the TLAR(1) process is described. The maximum likelihood estimator is found using a numerical iteration scheme. The properties of the estimator are investigated and compared to the least squares estimator using simulation.

For the TLAR(1) models ($\beta_1 = 1$ or $\beta_1 = -1$), (II.D.4.20) can be written as a one-dimensional function of the a variable α. We have

$$L(\alpha) = -n(\ell n2) - |x_1| + \sum_{i=2}^{n} \ell n\left\{ \frac{\alpha_1}{\sqrt{1-\alpha_1}} \exp\left\{ \frac{-|\nu_i|}{\sqrt{1-\alpha_1}} \right\} \right.$$

$$\left. + \sqrt{1-\alpha_1} \exp\left\{ \frac{-|x_i|}{\sqrt{1-\alpha_1}} \right\} \right\}, \qquad \text{(II.D.4.23)}$$

where

$$\nu_i = \begin{cases} x_i - x_{i-1} & \alpha \geqq 0, \\ x_i + x_{i-1} & \alpha < 0, \end{cases} \qquad \text{(II.D.4.24)}$$

$$-1 < \alpha < 1 \quad \text{and} \quad \alpha_1 = |\alpha|.$$

Now $L(\alpha)$ is continuous everywhere in the open interval (-1,+1) and differentiable everywhere except at $\alpha = 0$. The expressions for $\frac{dL(\alpha)}{d\alpha}$ and $\frac{d^2L(\alpha)}{d\alpha^2}$ are lengthy and cumbersome to use; hence are not given here.

Examples of the likelihood curve are given in Figures II.D.4.15 - II.D.4.18. Each curve was generated from a sample of 100 from a simulated TLAR(1) process with the stated α_1 and β_1. It is easy to see the non-differentiable point at zero and how flat the curve is. To see that there is a maximum (annotated with x on the figure) in these

TLAR(1): LOG-LIKELIHOOD FUNCTION; $\alpha_1 = .5$ AND $B_1 = -1$ SSN = 100

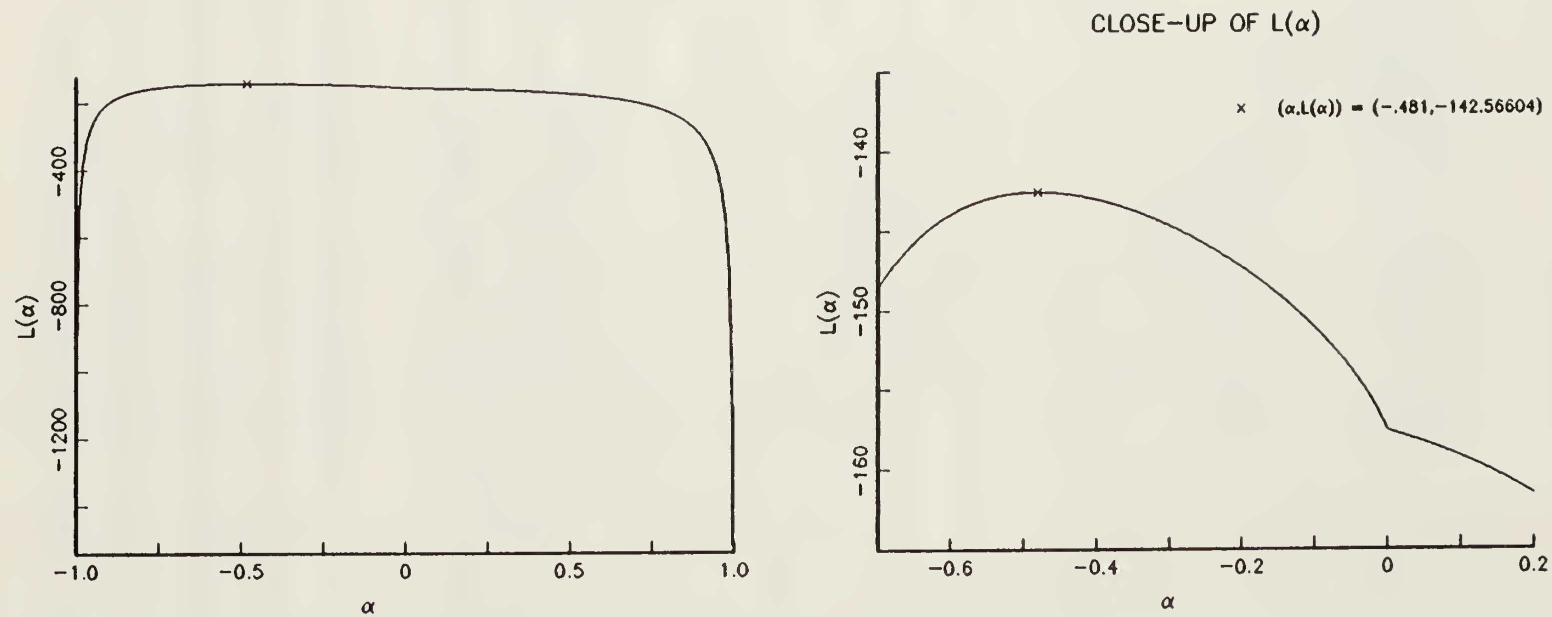


Figure II.D.4.15. TLAR(1): Log-Likelihood Function; α_1=.5, β_1=-1 and SSN=100

TLAR(1): LOG-LIKELIHOOD FUNCTION; α_1 = .1 AND B_1 = 1 SSN = 100

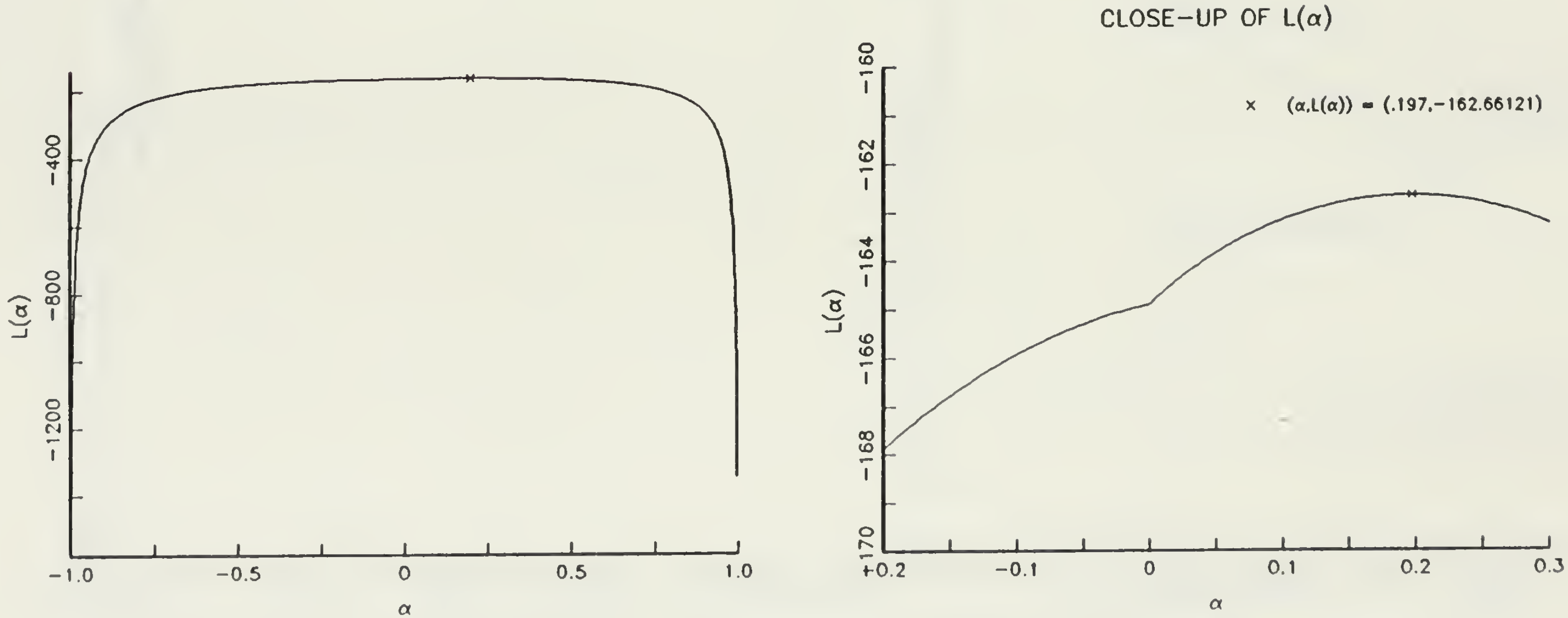


Figure II.D.4.16. TLAR(1): Log-Likelihood Function; α_1=.1, β_1=1 and SSN=100

TLAR(1): LOG-LIKELIHOOD FUNCTION; $\alpha_1 = .64$ AND $B_1 = 1$ SSN = 100

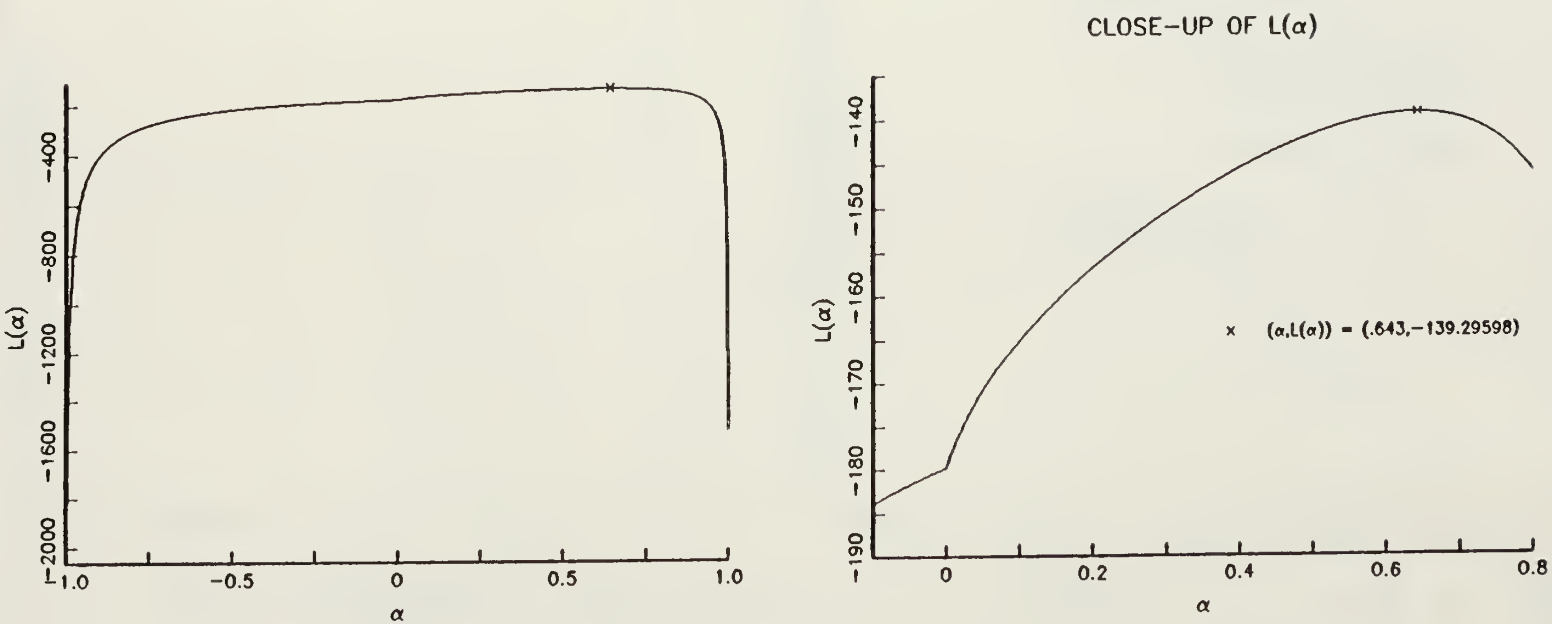


Figure II.D.4.17. TLAR(1): Log-Likelihood Function; α_1=.64, β_1=1 and SSN=100

TLAR(1): LOG-LIKELIHOOD FUNCTION; α_1 = .9 AND B_1 = 1 SSN = 100

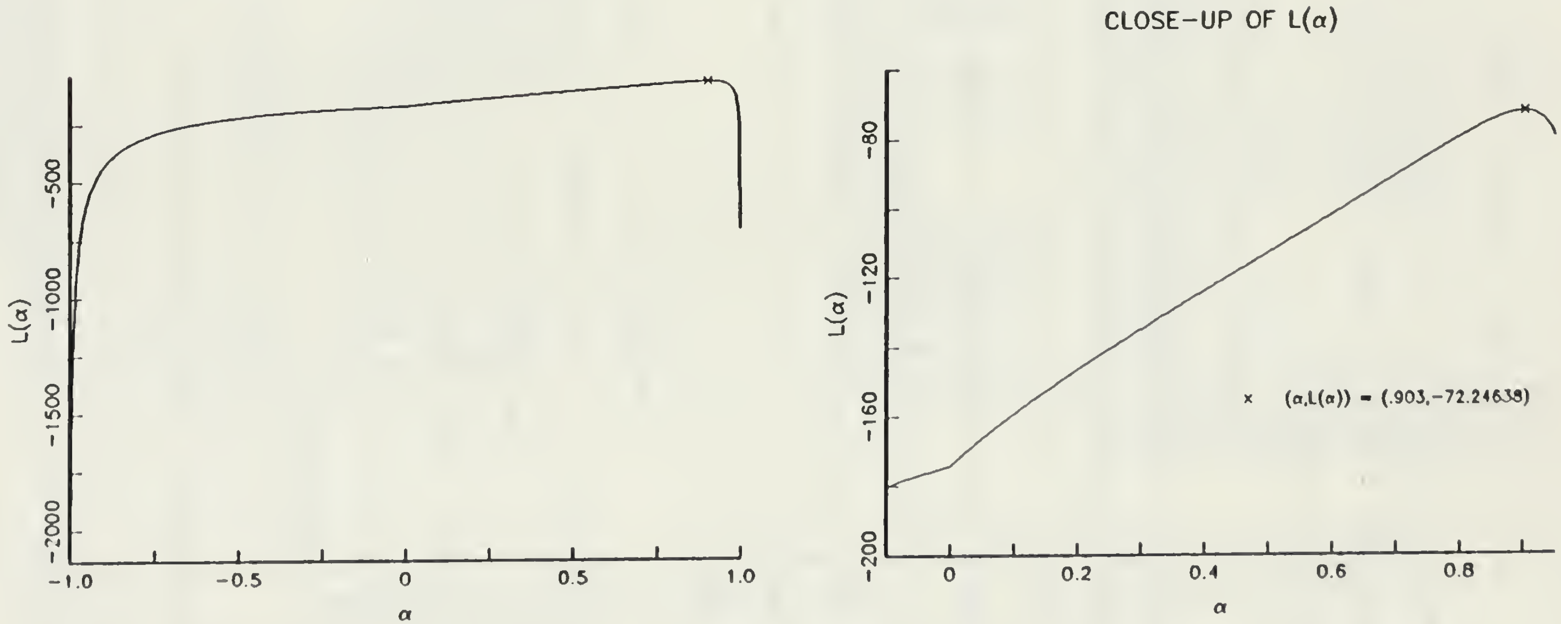


Figure II.D.4.18. TLAR(1): Log-Likelihood Function; α_1=.9, β_1=1 and SSN=100

curves, the second part of each figure focuses on the function near the true value of α_1.

The IMSL routine, ZXLSF, a one-dimensional search routine was used to find the value of α that maximized (II.D.4.23). The starting value α was the least squares estimator of serial correlation given by (II.D.4.15).

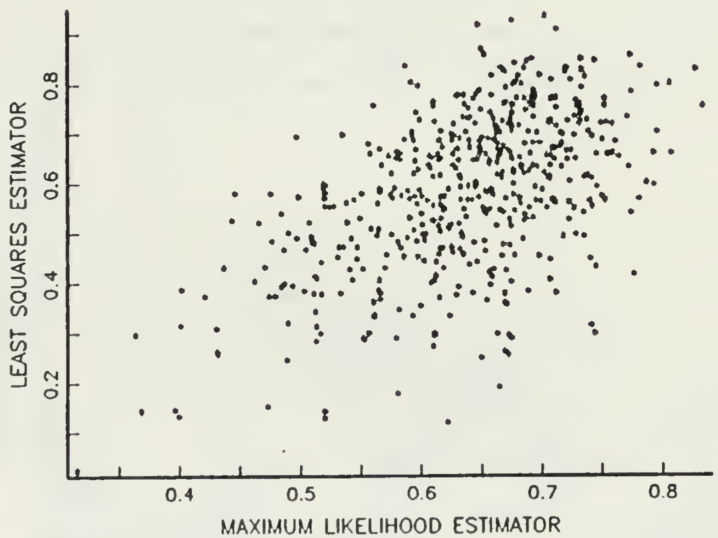
Using 500 samples of sizes 50 and 500, respectively, from simulated TLAR(1) processes with γ = .64, the scatter plot analyses in Figures II.D.4.19 and II.D.4.20 were completed. The least squares estimator and maximum likelihood estimator appear to be correlated. From the accompanying tables, the maximum likelihood estimator appears to have a smaller variance and bias than the least squares estimator. Analysis of the boxplots from a SIMTBED comparison of the least squares estimator and the maximum likelihoood estimator reflect the same results (see Figures (II.D.4.21 - II.D.4.22).

From the Normal plots given in Figure II.D.4.23, both the least squares and the maximum likelihood estimator appear to be coverging to a Normal distribution. There are three or four outliers in the tail out of 500 points.

E. OTHER CASES OF THE NLARMA(p,q) MODEL

1. Introduction

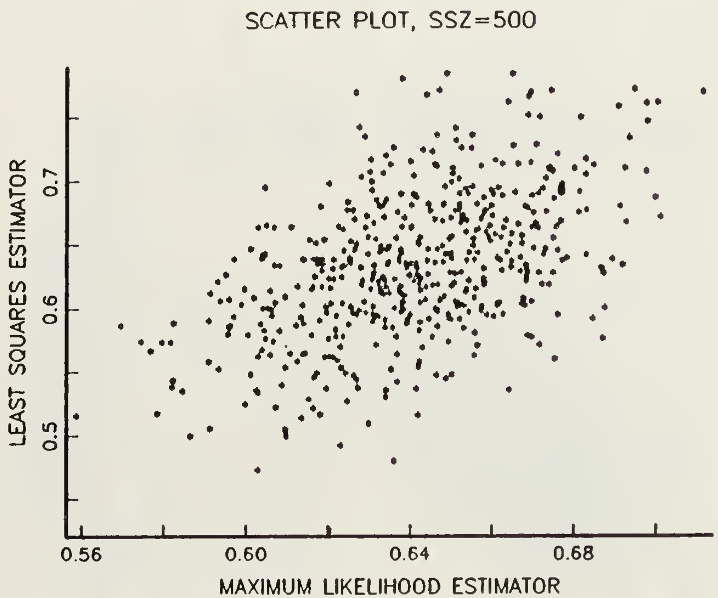
A primary advantage of the NLARMA(p,q) model is the ease with which the basic framework can be altered to cover a variety of different dependency structures. The NLAR(2) and NLAR(1) processes have been examined closely in the previous sections of this chapter. At this



SCATTER PLOT TABLE

X	:AMLE
Y	:ALS
SELECTION	:ALL
X LABEL	:MAXIMUM LIKELIHOOD ESTIMATOR
Y LABEL	:LEAST SQUARES ESTIMATOR
NO. OF ELEMENTS	:500
CORRELATION XY	:0.53072
RK CORRELATION	:0.49117 T=12.583
X MEAN	:0.63634
STD. DEVIATION	:0.082215
5-PERCENTILE	:0.48597
25-PERCENTILE	:0.5923
MEDIAN	:0.64776
75-PERCENTILE	:0.6911
95-PERCENTILE	:0.75128
X MIN.	:0.31515 0.3635 0.3685
X MAX.	:0.83299 0.82673 0.80681
Y MEAN	:0.58159
STD. DEVIATION	:0.15703
5-PERCENTILE	:0.29745
25-PERCENTILE	:0.48027
MEDIAN	:0.59131
75-PERCENTILE	:0.69501
95-PERCENTILE	:0.80943
Y MIN	:0.025289 0.11999 0.12931
Y MAX	:0.93753 0.92822 0.91911

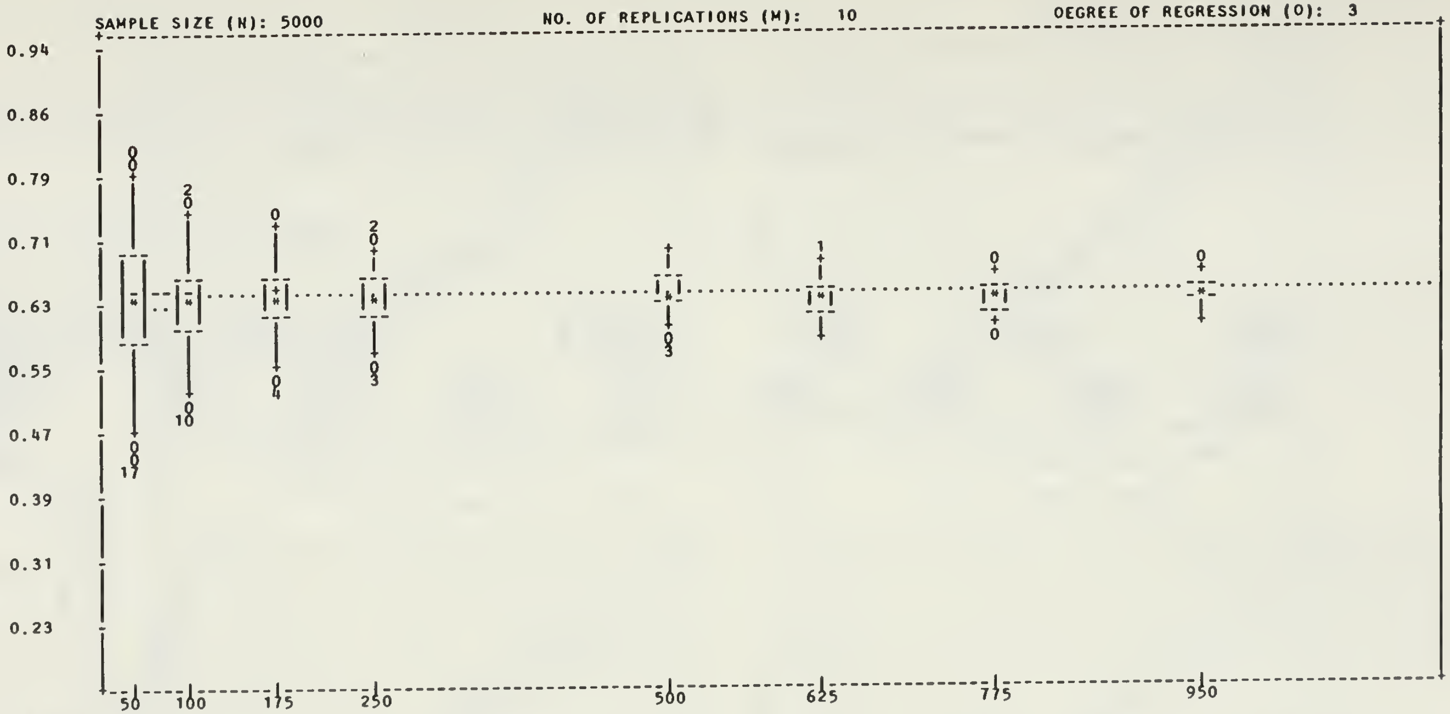
Figure II.D.4.19. Scatter Plot Analysis of the Maximum Likelihood and the Least Squares Estimators of γ in the TLAR(1) Process for 500 Samples of Size 50 with $\alpha_1=.64$ and $\beta_1=+1$



SCATTER PLOT TABLE

X	:AMLE1
Y	:ALS1
SELECTION	:ALL
X LABEL	:MAXIMUM LIKELIHOOD ESTIMATOR
Y LABEL	:LEAST SQUARES ESTIMATOR
NO. OF ELEMENTS	:500
CORRELATION XY	:0.50959
RK CORRELATION	:0.49277 T=12.637
X MEAN	:0.64081
STD. DEVIATION	:0.025833
5-PERCENTILE	:0.59702
25-PERCENTILE	:0.62336
MEDIAN	:0.64097
75-PERCENTILE	:0.65826
95-PERCENTILE	:0.68167
X MIN.	:0.55856 0.56954 0.57432
X MAX.	:0.71168 0.70108 0.70054
Y MEAN	:0.63372
STD. DEVIATION	:0.058426
5-PERCENTILE	:0.5363
25-PERCENTILE	:0.59584
MEDIAN	:0.63306
75-PERCENTILE	:0.67195
95-PERCENTILE	:0.73174
Y MIN	:0.4261 0.47358 0.48039
Y MAX	:0.78384 0.78356 0.78014

Figure II.D.4.20. Scatter Plot Analysis of the Maximum Likelihood and the Least Squares Estimators of γ in the TLAR(1) Process for 500 Samples of Size 500 with $\alpha_1=.64$ and $\beta_1=+1$

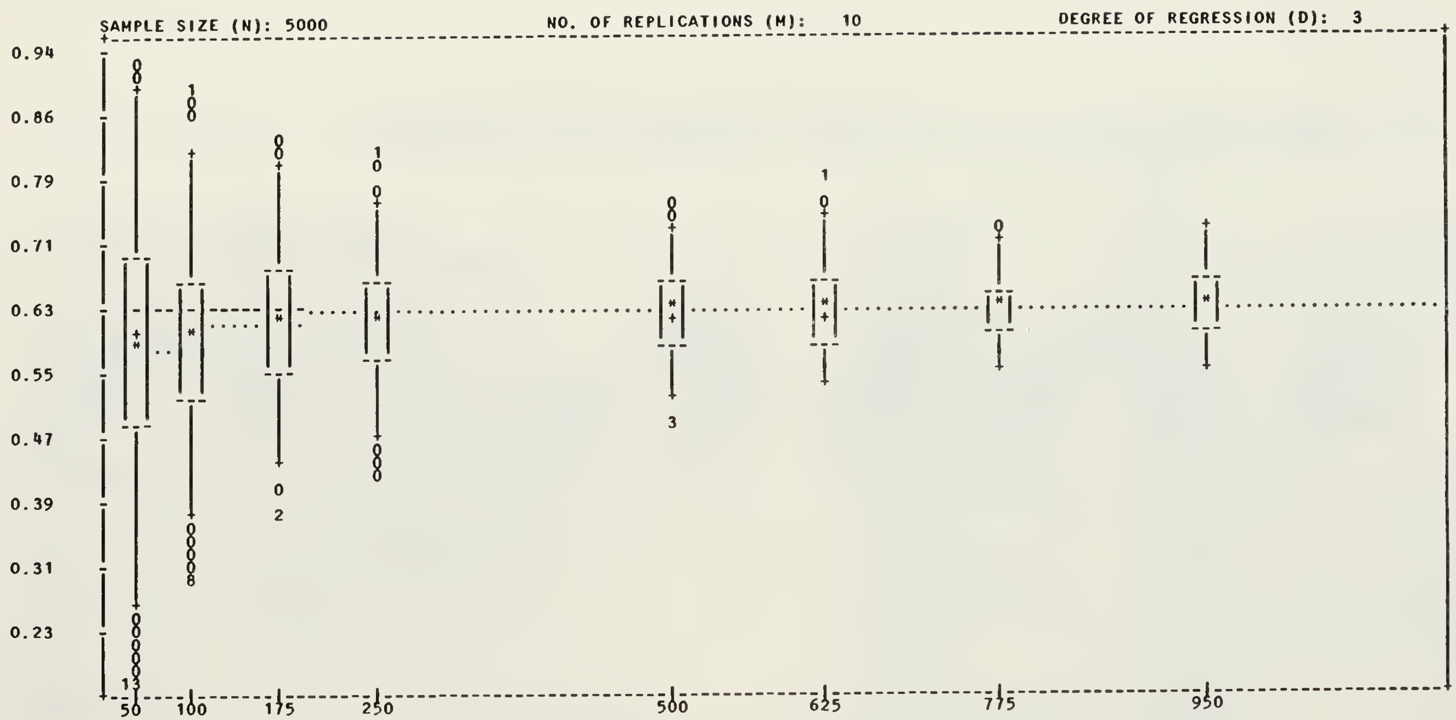


SUBSAMPLE SIZE	50	100	175	250	500	625	775	950
MEAN	0.6359	0.6340	0.6428	0.6366	0.6430	0.6381	0.6402	0.6432
STO	0.8272E-01	0.5908E-01	0.4306E-01	0.3505E-01	0.2675E-01	0.2340E-01	0.1846E-01	0.1827E-01
STD MEAN	0.2616E-02	0.2642E-02	0.2573E-02	0.2478E-02	0.2675E-02	0.2616E-02	0.2384E-02	0.2584E-02
SKEWNESS	-0.5605	-0.5013	-0.6512	-0.1146	-0.6099	0.1703	0.0007	-0.0008
KURTOSIS	0.5487	0.6185	1.0327	0.3095	0.8884	-0.0892	-0.1652	-0.1317

MEAN OF REGRESSION ON AVERAGES - COEFFICIENTS:	0.640421	0.635368	-195.278	7609.48
VARIANCE OF REGRESSION - COEFFICIENTS:	0.608232E-05	5.38994	129346.	0.154077E+09
STO OEV OF REGRESSION - COEFFICIENTS:	0.246624E-02	2.32162	359.647	12412.7
REGRESSION ON VARIANCE - COEFFICIENTS:	0.393038	-3.69587	53.3776	-210.580

ESTIMATOR: MAXIMUM LIKELIHOOD ESTIMATOR OF ALPHA IN TLAR(1) PROCESS AL=.64 BETA = 1.0 RHO= 0.64000
VERTICAL SCALE: YMIN = 0.1720
YMAX = 0.9433

Figure II.D.4.21. SIMTBED Boxplot Analysis of the Maximum Likelihood Estimator of γ with γ=.64 in the TLAR(1) Process



SUBSAMPLE SIZE	50	100	175	250	500	625	775	950
MEAN	0.5891	0.5980	0.6191	0.6226	0.6330	0.6334	0.6287	0.6333
STD	0.1548	0.1157	0.9063E-01	0.7720E-01	0.5716E-01	0.5691E-01	0.3991E-01	0.4055E-01
STD MEAN	0.4894E-02	0.5174E-02	0.5416E-02	0.5459E-02	0.5716E-02	0.6363E-02	0.5153E-02	0.5734E-02
SKEWNESS	-0.4931	-0.4342	-0.1738	0.1119	-0.0330	0.4702	0.1626	0.1148
KURTOSIS	0.2254	0.2926	0.1144	0.2292	0.4073	-0.0074	-0.2863	-0.7912

MEAN OF REGRESSION ON AVERAGES - COEFFICIENTS:	0.633500	-0.346865	-545.135	22584.8
VARIANCE OF REGRESSION - COEFFICIENTS:	0.394715E-04	19.3923	335324.	0.357140E+09
STD DEV OF REGRESSION - COEFFICIENTS:	0.628264E-02	4.40367	579.072	18898.1
REGRESSION ON VARIANCE - COEFFICIENTS:	1.50022	8.26411	-164.374	642.449

ESTIMATOR: LEAST SQUARES USING LINEAR RESIDUAL FROM TLAR(1) RANDOM COEFFICIENT PROCESS; RHO = 0.64000.
VERTICAL SCALE: YMIN = 0.1720
YMAX = 0.9433

Figure II.D.4.22. SIMTBED Boxplot Analysis of the Least Squares Estimator of γ with γ=.64 in the TLAR(1) Process

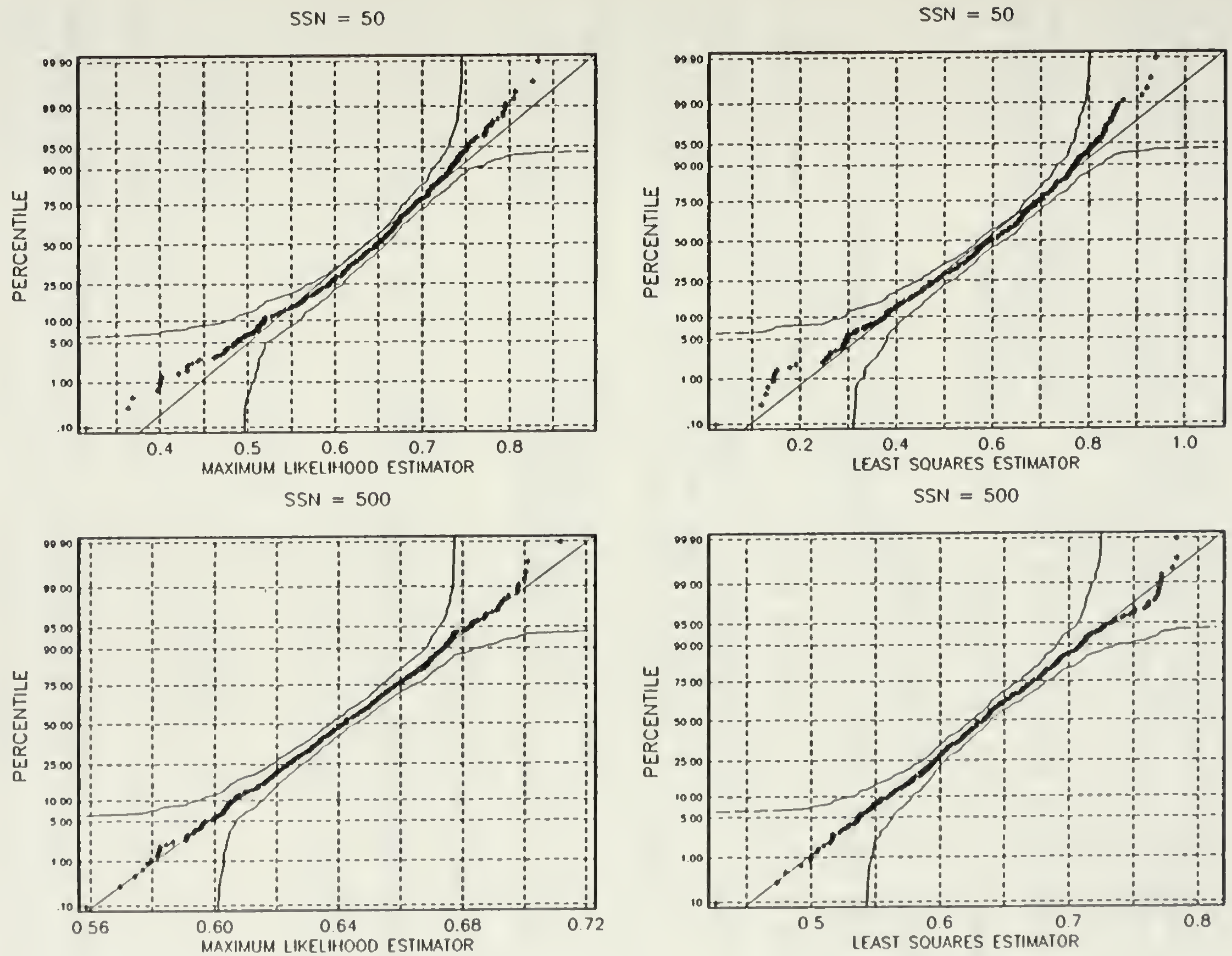


Figure II.D.4.23. Normal Probability Plots of the Maximum Likelihood and the Least Squares Estimators of γ in the TLAR(1) Process for Samples of Sizes 50 and 500 with γ=.64

time, the moving average first-order model, NLMA(1), and the mixed model, NLARMA(1,1), are briefly considered. The correlation structure and parameter space are discussed for each model.

The TLAR(1) model for which the maximum likelihood estimation was completed, can be easily extended. As the final part of this section, we present the p^{th}-order autoregressive processes for arbitrary $p \geqq 2$. The conditions for existence and uniqueness, the correlation structure and likelihood function are given. The maximum likelihood estimation scheme for the p parameters is also discussed.

2. A Backwards MA(1) Model, NLMA(1)

a. Correlation Structure of the NLMA(1) Process

From (II.D.1.1), we see that X_n is the random coefficient sum of independent variables each of which have a marginal Laplace distribution. Therefore, we can replace X_{n-1} by another Laplace variable. If it is independent of L_n and has a standard Laplace marginal distribution, then by the construction, X_n will still have a standard Laplace marginal distribution.

If we replace X_{n-1}, in fact, by L_{n-1} in (II.D.1.1), we obtain the following expression for X_n

$$X_n = K'_n \beta_1 L_{n-1} + K_n L_n , \qquad \text{(II.E.2.1)}$$

where $\{K'_n\}$ and $\{L_n\}$ are as given in (II.D.1.2) and $\{K_n\}$ is the corresponding two-valued discrete variable as given in (II.C.2.4) for the NLAR(2) model.

Since X_{n-k} is by construction in (II.E.2.1) independent of X_n for $|k| \geq 2$, we see that the model has the cut off property of a linear MA(1) model. The maximum range of correlations in any MA(1) is less than or equal to $|1/2|$, (Fuller [Ref. 29: p. 62]). This range is achieved by the linear MA(1) models. Some of the random coefficient MA(1) models have been shown to have a maximum range for the $\mathrm{Corr}(X_n, X_{n-1})$ to be strictly less than one-half (see Hugus [Ref. 30]).

Using (II.E.2.1) recursively, we derive the serial correlation in NLMA(1) as

$$\mathrm{Corr}(X_n, X_{n-1}) = \frac{\mathrm{Cov}(X_n, X_{n-1})}{\mathrm{Var}(X_n)},$$

$$= \frac{E\{(K'_n \beta_1 L_{n-1} + K_n L_n) X_{n-1}\}}{2},$$

$$= \frac{\alpha_1 \beta_1 E(L_{n-1} X_{n-1})}{2},$$

$$= \frac{\alpha_1 \beta_1}{2} E\{L_{n-1}(K'_{n-1} \beta_1 X_{n-2} + K_{n-1} L_{n-1})\},$$

$$= \alpha_1 \beta_1 E(K_{n-1}). \qquad \text{(II.E.2.2)}$$

Substituting in the values of the i.i.d. sequence $\{K_n\}$ with the corresponding probabilities p_2, $1-p_2$ from (II.D.1.3) we have

$$\mathrm{Corr}(X_n, X_{n-1}) = \alpha_1\beta_1\{(1-p_2) + \sqrt{(1-\alpha_1)\beta_1^2}\, p_2\}$$

$$= \alpha_1\beta_1\left\{\frac{1-\beta_1^2 + \alpha_1\beta_1^2\sqrt{(1-\alpha_1)\beta_1^2}}{1-(1-\alpha_1)\beta_1^2}\right\} . \qquad \text{(II.E.2.3)}$$

Figure II.E.2.1 is a contour plot of the level curves for $\rho(1) = \mathrm{Corr}(X_n, X_{n-1})$. Notice that in this model, the correlation is restricted in range over that of the linear MA(1) models. Using the IMSL global constrained optimization routine, ZXMWD, with multiple starts, the extremes for lag-1 serial correlation are $|\rho(1)| \leq 0.4026$, occurring at $\alpha_1 = .903$ and $\beta_1 = \pm.690$. In Chapter III, we give a continuous random coefficient model with MA(1) correlation structure, Laplace marginal distribution, and the full range of correlations, i.e. $|\rho(1)| \leq 5$.

b. Invertibility in NLMA(1)

It is well known (Chatfield [Ref. 31, p. 43]) that if

$$X_n = Z_n + \beta_1 Z_{n-1}, \qquad \text{(II.E.2.4)}$$

is a linear MA(1) model, then substituting $(1/\beta_1)$ in for β_1 does not change the autocorrelation function. This implies that the linear MA(1) model is not uniquely determined by its autocorrelation function.

It is also well known (Chatfield [Ref. 31: p. 43]) that by successive substitution, the MA(1) model in (II.E.2.4) can be written as the infinite autoregression

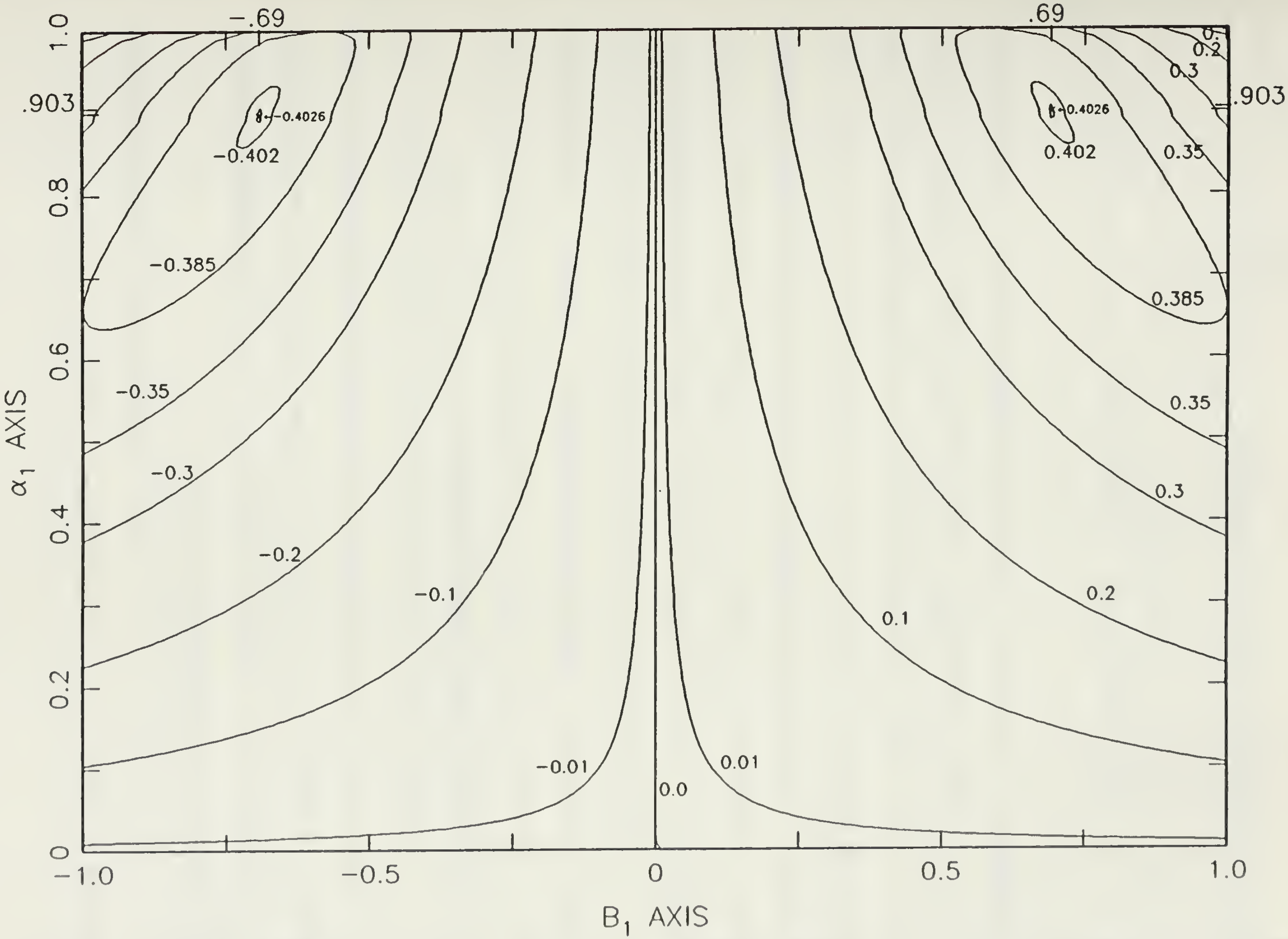


Figure II.E.2.1. NLMA(1): Contour Plot of the Feasible Region for $\rho(1)$ in Parameter Coordinates

$$Z_n = X_n - \beta_1 X_{n-1} + \beta_1^2 X_{n-2} - + \ldots. \qquad \text{(II.E.2.5)}$$

Likewise, if $1/\beta_1$ is in (II.E.2.4), we have

$$Z_n = X_n - \frac{1}{\beta_1} X_{n-1} + \frac{1}{\beta_1^2} X_{n-2} - + \ldots. \qquad \text{(II.E.2.6)}$$

Unfortunately, only one of the two processes given by (II.E.2.5) and (II.E.2.6) yields a convergent power series depending on whether $|\beta_1| < 1$ or not. Hence, the restriction on β_1 called "invertibility" by Box and Jenkins [Ref. 23: p. 50], guarantees a one-to-one correspondence between a linear MA(1) model and its autocorrelation function by restricting β_1 to be such that the MA(1) "inverted" infinite autoregression is the one with a convergent power series representation.

This definition of invertibility is not totally applicable to random coefficient models (such as NLMA(1)) with MA(1) correlation structure because it has not been established that there exists a corresponding infinite autoregression model.

Likewise, there can be an infinite number of models that have the same autocorrelation function and marginal distribution. This is the case in NLMA(1). As was seen in Figure II.E.2.1, each contour line corresponds to a constant value of $\rho(1)$ and is achievable by an infinite number of combinations of (α_1, β_1).

The purpose of this section then, is to find a different, but meaningful, way to restrict the (α_1, β_1) rectangle in Figure II.E.2.1

which: (1) does not further restrict the range of $\rho(1)$; and (2) which within the region the NLMA(1) model must be uniquely determined by $\rho(1)$ <u>and</u> either α_1 or β_1.

From the contours in Figure II.E.2.1, it appears that the feasible region for $\rho(1)$ can be partitioned in such a way that the two goals stated above can be achieved. It is not known, however, if this partition can be described analytically. Figure II.E.2.2 is an illustration of the partition into a center region and two complementary disjoint regions. The center region is roughly defined as the region to the right of a line from (-1,.667) to (-.577,1) and to the left of a line from (.577,1) to (1,.667). Both lines cut across the contours in the depression on the left and on the ridge on the right. The center region is more advantageous for two reasons. First, $\rho(1)$ is a continuous function of α_1 and β_1 in the center region. Secondly, the parameter estimation is more likely to be easier if the most extreme values of α_1 and β_1 can be avoided simultaneously. Therefore, we shall call the center region of Figure II.E.2.2 the "principal" region.

3. <u>A Mixed Autoregressive-Moving Average Model, NLARMA(1,1)</u>

From the theorem in Section II.C.2, we see that any two (possibly dependent) Laplace variables can be combined with an independent set of (again, possibly dependent) Laplace variables to form another Laplace variable. Using this property, if we replace X_{n-2} in NLAR(2) by L_{n-1}, then the marginal distribution of $\{X_n\}$ is still standard Laplace. We have then

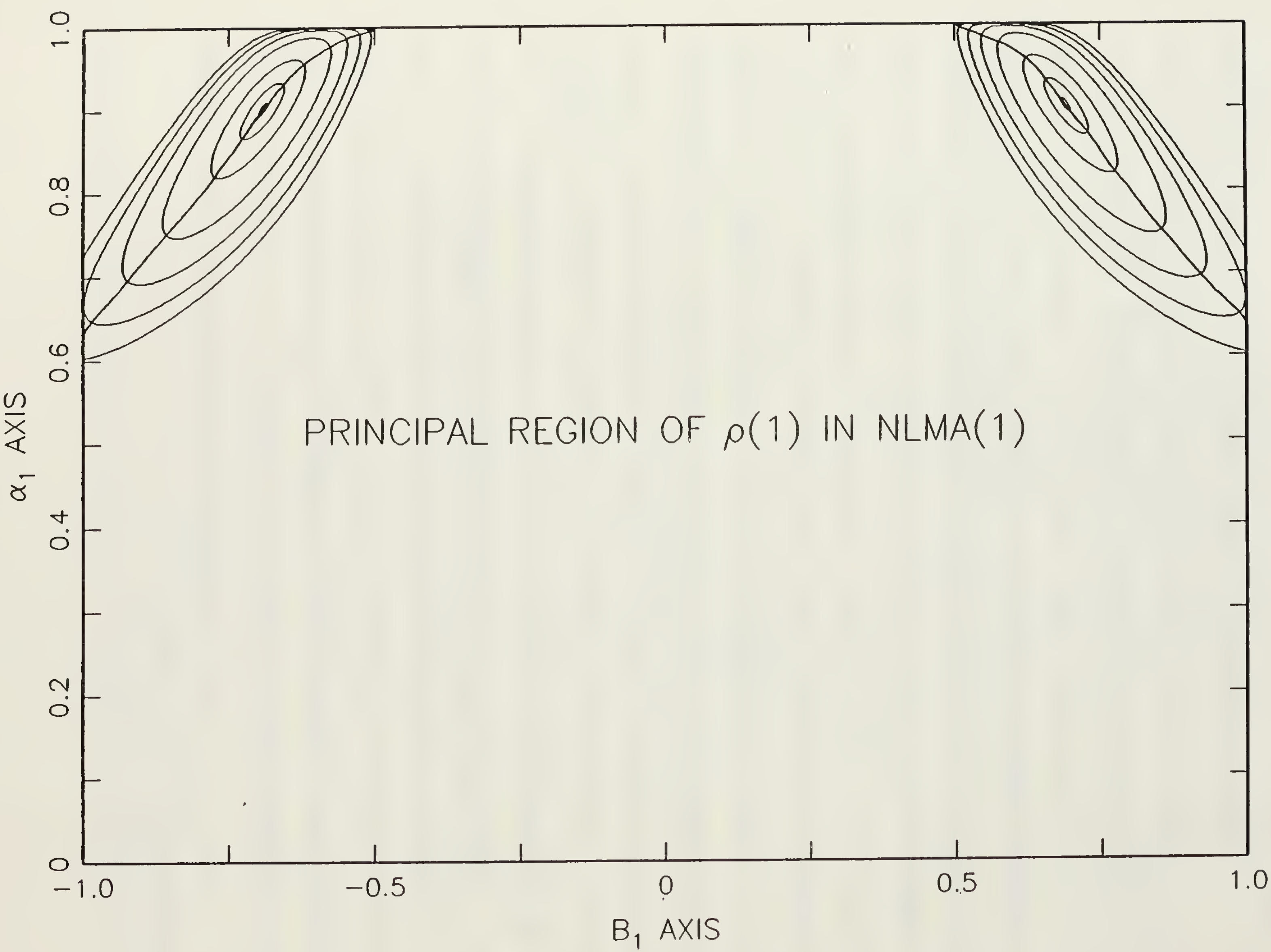


Figure II.E.2.2. NLMA(1): Boundary of Principal Region in Parameter Coordinates

$$X_n = \beta_1 K'_n X_{n-1} + \beta_2 K''_n L_{n-1} + K_n L_n, \qquad \text{(II.E.3.1)}$$

where $\{K'_n, K''_n\}$, $\{L_n\}$, $\{K_n\}$ are as previously defined.

Notice that if K'_n is identically zero, corresponding to $\alpha_1 = 0$, we obtain an expression of the form given by (II.E.2.1) for NLMA(1). Likewise, if K''_n is identically zero, we have the NLAR(1) model as given in (II.D.1.1).

The NLARMA(1,1) model has the same correlation structure as the linear mixed model ARMA(1,1). Using (II.E.3.1),

$$E(X_n X_{n-1}) = \alpha_1\beta_1 E(X^2_{n-1}) + \alpha_2\beta_2 E(L_{n-1}X_{n-1})$$
$$+ E(X_{n-1}K_n L_n). \qquad \text{(II.E.3.2)}$$

But X_{n-1}, K_n and L_n are independent so

$$E(X_n X_{n-1}) = 2\alpha_1\beta_1 + \alpha_2\beta_2\{\alpha_1\beta_1 E(L_{n-1}X_{n-2})$$
$$+ \alpha_2\beta_2 E(L_{n-1}L_{n-2}) + E(L^2_{n-1}K_{n-1})\}. \qquad \text{(II.E.3.3)}$$

Conditioning on K_{n-1}, using the independence of $\{L_n\}$ and (X_{n-2}, L_{n-1}) and dividing by the $\mathrm{Var}(X_n)$ we have

$$\rho(1) = \alpha_1\beta_1 + \alpha_2\beta_2(1-p_2-p_3+|b_2|p_2+|b_3|p_3)\ , \qquad \text{(II.E.3.4)}$$

where p_2, p_3, $|b_2|$, $|b_3|$ are defined in (II.C.2.5) through (II.C.2.9). For $\ell \geqq 2$, (II.E.3.3) and (II.E.3.4) become

$$E(X_n X_{n-\ell}) = \alpha_1 \beta_1 E(X_{n-1} X_{n-\ell}) \qquad \text{(II.E.3.5)}$$

and

$$\rho(\ell) = \alpha_1 \beta_1 \rho(\ell-1). \qquad \text{(II.E.3.6)}$$

These equations are the same as those of the ARMA(1,1) model (see Chatfield [Ref. 36: p. 58]). However, the range of correlations is significantly reduced over that of ARMA(1,1). Figure II.E.3.1 represents a side-by-side comparison of the $(\rho(1), \rho(2))$ space for NLARMA(1,1) and the familiar linear ARMA(1,1). Although $\rho(1)$ can range from -1 to +1, the combinations with $\rho(2)$ are severely limited in NLARMA(1,1). The minimum $\rho(2)$ in NLARMA(1,1), found numerically using the reduced gradient method is approximately -.025 at $\rho(1) \approx \pm.2$. As $|\rho(1)|$ increases, $\rho(2)$ approaches $\rho(1)^2$.

4. <u>Higher Order Autoregressive Models, TLAR(p)</u>

a. Introduction

It has been stated by Raftery [Ref. 32] that there exists NEAR(p) models for $p \geqq 2$. Also, Nicholls and Quinn [Ref. 16] have given conditions for the existence and uniqueness, strict stationary, etc., for general RCA(p) models. However, only for the NEAR(2) and the NLAR(2) processes has it been shown explicitly what the necessary innovation is; and that it is a valid random variable.

POINT PLOTS OF ADMISSIBLE REGION FOR $\rho(1)$ AND $\rho(2)$

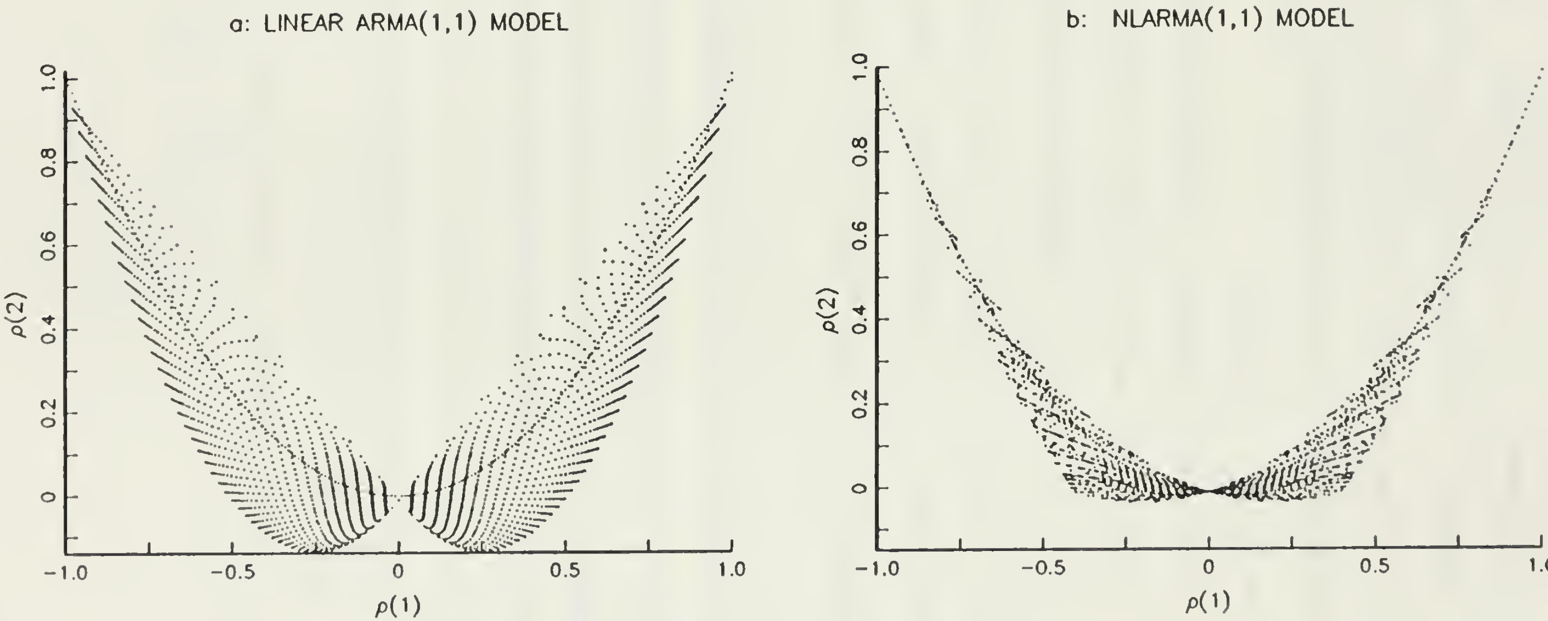


Figure II.E.3.1. Point Plots of Admissible Region for $\rho(1)$ and $\rho(2)$ for Linear ARMA(1,1) and NLARMA(1,1) Processes

For $p \geqq 3$, this has not been accomplished for the general NEAR(p) process; nor is it done now for the NLAR(p) process. However, there are 2^p different p^{th} order autogressive models with p parameters that are special cases of the NLAR(p) process. These models are called the TLAR(p) models. The innovation for the second-order model was given without proof following the theorem in Section II.C.2. The likelihood function and maximum likelihood estimation of α_1 was given in Section II.D.4 for the TLAR(1) processes.

The TLAR(2) models, including the two TLAR(1) models only account for four of the infinite number of NLAR(2) models which all have the same AR(2) correlation structure and standard Laplace marginal distribution. Since there is a variety of different sample path behaviors obtainable in the general NLAR(2) model, it is possible that a TLAR(2) model will not always be the most appropriate model for a given set of data.

However, as is shown in the remainder of this section, the TLAR(p) models have an advantage over the general NLAR(p) models. The TLAR(p) processes for $p \geqq 3$ exist; are easily constructed; are partially time reversible; and are parsimonious with respect to parameters. The parameters in the TLAR(p) process are easily estimated from the conditional likelihood function by the method of maximum likelihood.

b. Existence and Uniqueness

The TLAR(p) models $p \geqq 1$ have the form

$$X_n = \sum_{i=1}^{p} K_n^{(i)} X_{n-i} + \varepsilon_n, \qquad \text{(II.E.4.1)}$$

where $\{X_n\}$ is assumed stationary with Laplace marginal distribution; $\{K_n^{(1)},\ldots,K_n^{(p)}\} = K_n$ is a p-variate discrete random variable independent of $\{\varepsilon_n\}$ and $X_{n-1}, X_{n-2},\ldots$. For all n

$$K_n = \begin{cases} (1, 0, 0,\ldots, 0) & \text{w.p. } \alpha_1 \\ (0, 1, 0,\ldots, 0) & \text{w.p. } \alpha_2 \\ \vdots & \vdots \\ (0, 0, 0,\ldots, 1) & \text{w.p. } \alpha_p \\ (0, 0, 0,\ldots, 0) & \text{w.p. } 1 - \sum_{i=1}^{p} \alpha_i = \lambda > 0, \end{cases} \qquad \text{(II.E.4.2)}$$

so $E(K_n^{(i)}) = \alpha_i$ for all $i = 1,\ldots,p$. The 2^p choices of model arise from the selection of signs for each of the X_{n-i} (either +1 or -1).

Now if $\{X_n\}$ is stationary, then the following expression for the characteristic function of the i.i.d. innovation, ε_n, follows from (II.E.4.1) regardless of the choice of signs on X_{n-i}. (The distribution of a symmetric random variable Z is the same as that for -Z). We have,

$$\phi_{X_n}(\omega) = E[\exp\{-i\omega(\sum_{i=1}^{p} K_n^{(i)} X_{n-i} + \varepsilon_n)\}],$$

$$= \phi_{\varepsilon_n}(\omega) \; [\sum_{i=1}^{p} \alpha_i \; \phi_{X_{n-i}}(\omega) + \lambda],$$

$$= \phi_{\varepsilon_n}(\omega)\,[(1-\lambda)\,\phi_{X_n}(\omega) + \lambda],$$

from conditioning on K_n, the stationarity assumption of $\{X_n\}$ and the independence of ε_n of $X_{n-1}, X_{n-2}, \ldots$. Therefore, substituting from (II.B.1.2)

$$\phi_\varepsilon(\omega) = \left(\frac{1}{1+\omega^2}\right) / \left\{\frac{(1-\lambda)}{1+\omega^2} + \lambda\right\}$$

$$= 1/(1+\lambda\omega^2). \qquad \text{(II.E.4.3)}$$

For $\lambda > 0$, (II.E.4.3) is recognized as the characteristic function of a scaled Laplace random variable with scale parameter $\sqrt{\lambda}$.

Since (II.E.4.1) can be written as

$$X_n = \sum_{i=1}^{p} \{\alpha_i X_{n-i} + (K_n^{(i)} - \alpha_i) X_{n-i}\} + \varepsilon_n \qquad \text{(II.E.4.4)}$$

and satisfies the conditions in Section II.C.2, the TLAR(p) models are RCA(p) models. Since the innovation $\{\varepsilon_n\}$ and $\{K_n\}$ are i.i.d., then TLAR(p) are strictly stationary and $\{X_n\}$ is the unique solution by the theorems of Nicholls and Quinn [Ref. 16: p. 31 and p. 37].

c. Correlation Structure

The TLAR(p) models are p^{th}-order autoregressive in the sense that $E(X_n | X_{n-1} = x_1, X_{n-2} = x_2, \ldots, X_{n-p} = x_p)$ is a linear function in x_i, $i = 1, \ldots, p$. It is also autoregressive in the sense that it

satisfies a set of Yule-Walker equations. Multiplying (II.E.4.1) by $X_{n-\ell}$, $\ell \geq 1$, and taking expectations, we have

$$E(X_n X_{n-\ell}) = a_1 E(X_{n-1} X_{n-\ell}) + \ldots + a_p E(X_{n-p} X_{n-\ell}). \qquad \text{(II.E.4.5)}$$

Dividing by $\mathrm{Var}(X_n)$ and substituting $\ell = 1,\ldots,p$ into (II.E.4.5), we have the set of equations

$$\begin{aligned} \rho(1) &= a_1 + a_2\rho(1) + \ldots + a_p\rho(p-1) \\ \rho(2) &= a_1\rho(1) + a_2 + \ldots + a_p\rho(p-2) \\ &\vdots \\ \rho(p) &= a_1\rho(p-1) + a_2\rho(p-2) + \ldots + a_p, \end{aligned} \qquad \text{(II.E.4.6)}$$

where $a_i = \alpha_i(\text{Sign of } X_{n-i})$ for all $i = 1,\ldots,p$.

For the TLAR(2) cases, the $(\rho(1), \rho(2))$ admissible region is the entire diamond given in Figure II.C.3.1. It is divided, however, into four right triangles, one per quadrant, corresponding to the sign of X_{n-1} and X_{n-2} in the model.

d. Conditional Density of $X_n | X_{n-1}, X_{n-2}, \ldots, X_{n-p}$

The conditional density for each of the 2^p specific choices of signs are easily found noting that the conditional probability is just a sum of Laplace cumulative distributions. We have

$$P(X_n < x | X_{n-1} = x_1, X_{n-2} = x_2, \ldots, X_{n-p} = x_p)$$

$$= P(K_n^{(1)} x_1 + K_n^{(2)} x_2 + \ldots + K_n^{(p)} x_p + \sqrt{\lambda}\, L_n < x)$$

$$= \alpha_1 P(\sqrt{\lambda}\, L_n < x - x_1) + \ldots + \alpha_p P(\sqrt{\lambda}\, L_n < x - x_p) + \lambda P(\sqrt{\lambda}\, L_n < x)$$

$$= \alpha_1 F_L\left\{\frac{x-x_1}{\sqrt{\lambda}}\right\} + \ldots + \alpha_p F_L\left\{\frac{x-x_p}{\sqrt{\lambda}}\right\} + \lambda F_L\left\{\frac{x}{\sqrt{\lambda}}\right\}, \qquad \text{(II.E.4.7)}$$

where $F_L(\cdot)$ is the cumulative distribution function of a standard Laplace random variable. Taking derivatives with respect to x, we have

$$f_{X_n | X_{n-1}, \ldots, X_{n-p}}(x | x_1, \ldots, x_p) = \frac{1}{\sqrt{\lambda}} \sum_{i=1}^{p} \alpha_i f_L\left\{\frac{x-x_i}{\lambda}\right\} + \sqrt{\lambda}\, f_L\left\{\frac{x}{\sqrt{\lambda}}\right\}. \qquad \text{(II.E.4.8)}$$

e. Conditional Maximum Likelihood Estimation of $(a_1, \ldots, a_p)$

Since there are many p-variate Laplace distributions that $(X_p, \ldots, X_1)$ could have, and that the particular one is not known to us, it is not possible to form the exact likelihood function which is written

$$f_{X_n \ldots X_1} = \left\{ \prod_{i=p+1}^{n} f_{X_i | X_{i-1}, \ldots, X_{i-p}} \right\} f_{X_p, \ldots, X_1}. \qquad \text{(II.E.4.9)}$$

Instead, we can calculate the conditional log-likelihood function as the logarithm of the product of the first (n-p) terms in

(II.E.4.9). This is commonly done. See, for example, Priestly [Ref. 33: p. 350]. Using $a_i = \alpha_i \text{sign}(X_{n-i})$, we have the following single expression for the conditional log-likelihood function, given the n realizations from TLAR(p) process, written as a function of a_i for $i = 1,\ldots,p$,

$$L(a_1,\ldots,a_p) = \sum_{i=p+1}^{n} \ell n \left\{ f_{X_i | X_{i-1},\ldots,X_{i-p}} \right\}$$

$$= \sum_{i=p+1}^{n} \ell n \left\{ \frac{1}{\sqrt{\lambda}} \left\{ \sum_{j=1}^{p} \alpha_j f_L\left(\frac{v_{ij}}{\sqrt{\lambda}} \right) \right\} + \sqrt{\lambda}\, f_L\left(\frac{x_i}{\sqrt{\lambda}} \right) \right\}, \qquad \text{(II.E.4.10)}$$

where

$$v_{ij} = \begin{cases} x_i - x_{i-j} & \text{if } a_j \geq 0, \quad j = 1,\ldots,p, \\ x_i + x_{i-j} & \text{if } a_j < 0, \end{cases} \qquad \text{(II.E.4.11)}$$

$i = p+1,\ldots,n$; $\alpha_j = |a_j|$ and λ are functions of the variable a_j.

We see that when $p = 1$ (II.E.4.10) and (II.E.4.11) give the expressions used in the TLAR(1) process in Section II.D.4.

As a function of $(a_1,\ldots,a_p)$, (II.E.4.10) is continuous throughout the interior of the p-dimensional subspace on which it is defined. It is not differentiable with respect to a_i anywhere that $a_i = 0$. The maximization of (II.E.4.10) can be formulated as a constrained non-linear program for which a numerical routine would

probably be required to solve for $(\hat{a}_1,...,\hat{a}_p)$, the joint conditional likelihood estimator of $(a_1,...a_p)$.

III. CONTINUOUS RANDOM COEFFICIENT MODELS WITH SYMMETRIC NON-NORMAL MARGINALS

A. INTRODUCTION

The discrete random coefficient NLARMA(p,q) models studied in the previous chapter offered a variety of different dependency structures analogous to their linear ARMA(p,q) counterparts as described in the Box-Jenkins approach to time series analysis. These models, however, could be considered deficient in some ways. For one thing, all the models have, by design, the same marginal distribution, i.e. Laplace. To obtain a different marginal distribution would require starting over to develop the appropriate innovation sequence. Raftery [Ref. 32] has reported some results in extending the NEAR framework to other models with different marginals and ARMA correlation structures.

Furthermore, the parameter estimation, which is easy to do in Gaussian linear AR(p) models, is not particularly easy in the autoregressive process of the NLARMA(p,q) family. In the moving average and mixed models of NLARMA(p,q), the maximum likelihood procedure is even more difficult. Raftery [Ref. 32] claims that the maximum likelihood estimator of β_1 in the NLAR(1) process would be super-efficient based on his work in parameter estimation in the NEAR(1) process and the extensions that he has proposed. Super-efficiency is not an attractive property of an estimator.

Again, the moving average model, NLMA(1), does not allow for the full range of correlations that are obtainable with the linear MA(1) model.

Finally, note that there is another attractive property of the random coefficient models that is not fully exploitable in the discrete random coefficient models (NEAR(1) and NLAR(1)). That is, in the NLARMA(p,q) models the coefficients of the process can change somewhat over time and the process itself remains stationary. Andel [Ref. 34] has noted that in many applications of time series analysis, particularly in the fields of hydrology, meteorology and biology, the coefficients of the model are attempting to describe complicated processes. The coefficients may have some random behavior of their own, apart from that usually attributed to the independent innovation sequence.

If stationary constant coefficient models are not particularly good at modelling such systems (as suggested by Andel [Ref. 34]), then the NLARMA(p,q) models would not be much better because the coefficients are limited to a finite (very small) number of possible values. However, Lawrance and Lewis [Ref. 6] have shown in the case of NEAR(1) that it is possible to alter the character of the sample paths of a given low-order autoregression by extending the two-parameter model to one having 4 parameters. The number of extra parameters could be excessive and the costs in parameter estimation unacceptable.

In this chapter, a different family of stationary random coefficient time series models is introduced which retains many of the favorable aspects of the NLARMA family (specified marginal and correlation

structure) and offers alternatives in the areas pointed out above as disadvantages in the NLARMA construction.

The symmetric marginal distribution can be specified by one shape parameter to be any one of an infinite number of non-Gaussian distributions. This family is the ℓ-Laplace family and is examined in the next section. The family--including as a special case the double exponential (Laplace) distribution--has members with extremely high kurtosis, as well as those that have a limiting kurtosis that approaches that of the Gaussian distribution. This offers a significant advantage over the NLARMA models.

Just as discrete random variables are needed for the coefficients in the NLARMA(p,q) models, the square roots of Beta random variables are used in this family of models to maintain the ℓ-Laplace marginal distribution. The square root Beta transformation theorem is the key result through which all the time series models in this chapter are formulated. By the theorem, Laplace variables are changed into those that have ℓ-Laplace distributions. Previous uses of Beta random variables in modelling non-Normal time series is evident in the models with Gamma marginals of Lewis [Ref. 35] and Hugus [Ref. 30].

The fact that the coefficients are continuous instead of discrete allows for a continuous variation. That they are functions of Beta random variables restricts the variation to a bounded interval. This is likely to facilitate the modelling of those "complicated" systems as described by Andel [Ref. 34].

The principal models investigated in this chapter are those with first-order autoregressive correlation structure. They are first-order Markov processes. For the purpose of discussing parameter estimation in this family of autoregressive models, as opposed to the NLAR(1) family, the focus is narrowed to that AR(1) model of the family with Laplace marginals--the so-called Beta-Laplace First-Order Autoregressive model, BELAR(1). Several point estimators of location and scale are discussed and examined through simulation in SIMTBED [Ref. 15]. The one parameter which uniquely determines all the correlations of lag k in the BELAR(1) model can be estimated by a least squares procedure which has very nice asymptotic properties. The maximum likelihood estimator of serial correlation is also obtained using numerical methods.

First-order autoregressive correlation structure is not the only type of dependency relationship that is obtainable from using the square root Beta-Laplace transformation. In the last section of this chapter a first-order moving average model and an extension to a q^{th}-order model are introduced. The MA(1) model retains the full range of correlations of the linear MA(1) models. This was not the case in the NLMA(1) model.

B. ℓ-LAPLACE DISTRIBUTION

1. The ℓ-Laplace Random Variable

It was shown in Section II.B that the standard Laplace distribution belongs to the class of infinitely divisible distributions. The probability density function of a Laplace distributed variable was given in (II.B.1). The characteristic function of the standard Laplace random variable was given in (II.B.2). Thus if

$$\phi_X(\omega) = \left(\frac{1}{1+\omega^2}\right)^{\ell}, \quad \ell > 0, \tag{III.B.1.1}$$

then X is a random variable. In fact it is the difference of two independent, identically distributed Gamma(ℓ,1) random variables where ℓ is the shape parameter and 1 is the scale parameter. Therefore, if X has a characteristic function given by (III.B.1.1), then X is an <u>ℓ-Laplace</u> random variable.

Since (III.B.1.1) is a real function of ω, X is a symmetric random variable. It is easily verified that

$$E(X^n) = \begin{cases} 0 & \text{if } n \text{ is odd,} \\ (k+1)^{[k]}\ell^{[k]} & \text{if } n = 2k, \ k = 1,2,\ldots, \end{cases} \tag{III.B.1.2}$$

where $b^{[k]} = b(b+1)\ldots(b+k-1)$ for all $b > 0$. Since all odd moments are zero in (III.B.1.2), the ℓ-Laplace distribution is not skewed for any $\ell > 0$. From (III.B.1.2) we find that the kurtosis is

$$\gamma_2 = \frac{E(X_n^4) - (E(X_n))^4}{\mathrm{Var}^2(X_n)} = \frac{3^{[2]}\ell^{[2]}}{(2\ell)^2} = 3 + \frac{3}{\ell}. \tag{III.B.1.3}$$

The kurtosis approaches 3 as $\ell \to \infty$, which corresponds to that of a Normal distribution.

Since an ℓ-Laplace random variable, $X(\ell)$, is the difference of two i.i.d. Gamma(ℓ,1) random variables, we obtain the density for $X(\ell)$ by using conditional expectations.

If $G_1(\ell,1)$ and $G_2(\ell,1)$ are the i.i.d. Gamma$(\ell,1)$ random variables, then conditioning on $G_2(\ell,1)$, we have

$$\begin{aligned} P(X<x) = P(G_1-G_2<x) &= E_{G_2}\{P(G_1-G_2<x|G_2=g)\} \\ &= E_{G_2}\{P(G_1<x+g)\} \end{aligned} \tag{III.B.1.4}$$

Since Gamma random variables are non-negative,

$$P(G_1<x+g) = \begin{cases} 0 & \text{if } g \leqq -x, \\ F_{G_1}(x+g) & \text{if } g > x, \end{cases} \tag{III.B.1.5}$$

where $F_{G_1}(x+g)$ is the cumulative distribution function of G_1. The expressions are shortened from $G_i(\ell,1)$ to $G(\ell,1)$, because they are i.i.d. Therefore, (III.B.1.4) can be written as

$$P(X<x) = \int_{g=L(x)}^{g=\infty} F_G(x+g)f_G(g)dy, \tag{III.B.1.6}$$

where

$$f_G(g;\ell) = \begin{cases} \dfrac{g^{\ell-1}\exp(-g)}{\Gamma(\ell)} & g > 0, \\ 0 & \text{otherwise}, \end{cases} \tag{II.B.1.7}$$

is the density of a Gamma $(\ell,1)$ random variable and again, because of the non-negativity

$$L(x) = \begin{cases} 0 & \text{if } x \geq 0, \\ -x & \text{if } x < 0. \end{cases} \qquad \text{(III.B.1.8)}$$

Differentiating (III.B.1.6) using Leibniz' rule for the derivative of an integral with variable limits, we have, after some simplification

$$f_X(u;\ell) = \int_{g=L(u)}^{g=\infty} \frac{1}{\Gamma^2(\ell)} \left\{\frac{1}{g(g+u)}\right\}^{1-\ell} \exp\{-(2g+u)\}\, dg. \qquad \text{(III.B.1.9)}$$

Now if ℓ is a positive integer, (III.B.1.9) can be evaluated analytically using integration by parts. If $\ell = 1$ we obtain the density of the standard Laplace distribution. For $\ell = 2,3,4$ the densities are also well known derivations given, for example, as textbook problems by Feller [Ref. 25: p. 64]. Feller however looks at the results of (III.B.1.9) as the n-fold convolution ($n = 2,3,4$) of i.i.d. standard Laplace random variables. Figure III.B.1.1 shows the densities of the ℓ-Laplace random variable for $\ell = 1,2,3,4$. Note how the graphs take on the shape of a Normal density with $\sigma^2 = 2\ell$.

2. Numerical Evaluation of the ℓ-Laplace Density

If $\ell > 0$ and is not an integer, then (III.B.1.9) must be evaluated numerically. We will be interested in the evaluation of the density in (III.B.1.9) for $0 < \ell < 1$, in order to calculate conditional densities and likelihood function.

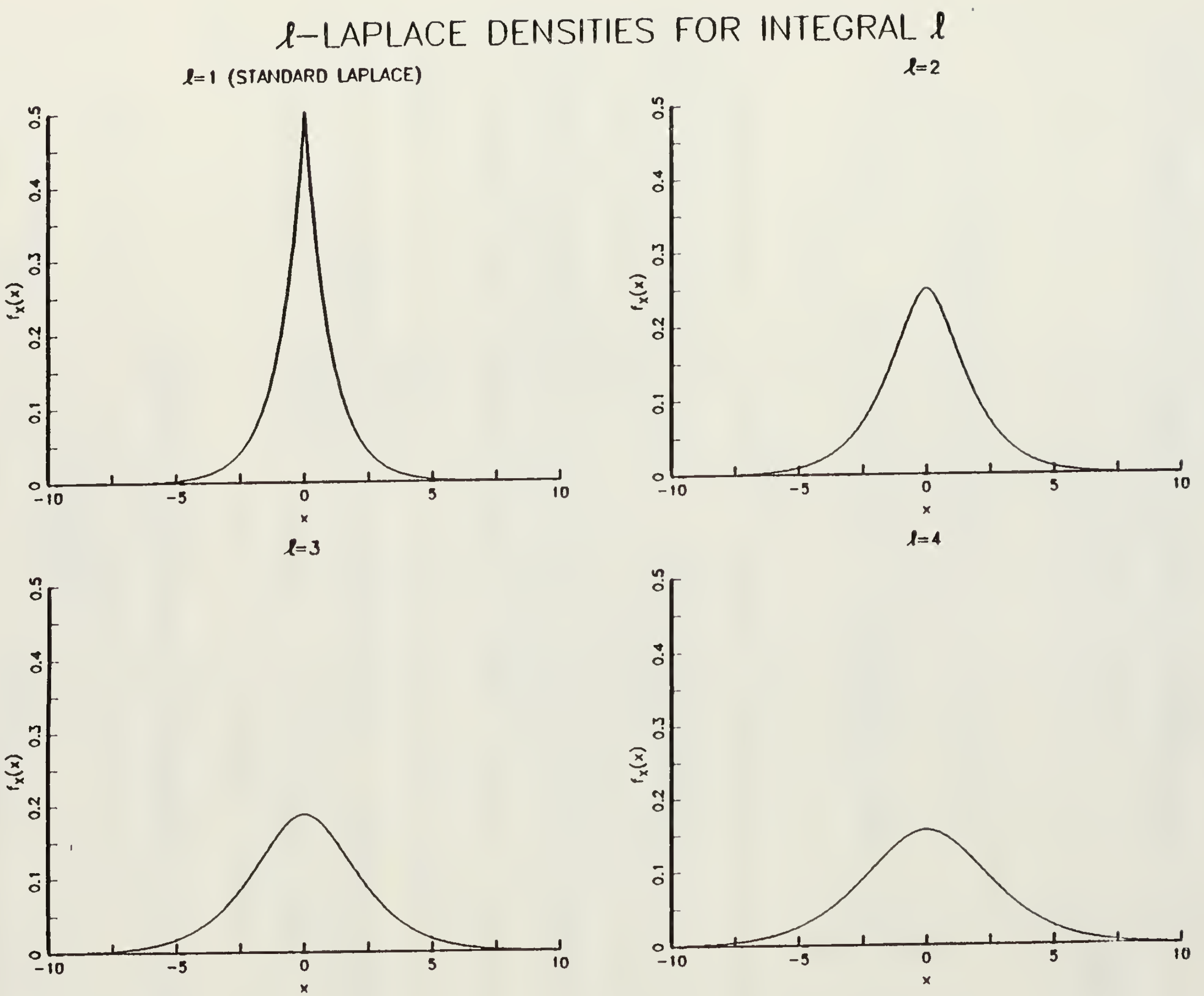


Figure III.B.1.1. Examples of the ℓ-Laplace Density for Integral Values of ℓ by Exact Evaluation of Equation III.B.1.9

Figure III.B.2.1 displays examples of densities for non-integral ℓ obtained by using the IMSL numerical integration scheme DCADRE to evaluate (III.B.1.9). The upper limit of integration in (III.B.1.9) is replaced by a suitable constant $m > 1$. Since for $g > 1$ and fixed ℓ and $u > 0$,

$$\frac{1}{\Gamma^2(\ell)}\left(\frac{1}{g(g+u)}\right)^{1-\ell} \exp\{-(2g+u)\} < \frac{\exp\{-(u+2g)\}}{\Gamma^2(\ell)}, \qquad \text{(III.B.2.1)}$$

then

$$|\text{DCADRE}-f_X(u;\ell)| < \frac{\exp\{-(u+2m)\}}{2\Gamma^2(\ell)}. \qquad \text{(III.B.2.2)}$$

Difficulty in integrating comes about because of the singularity at the lower limit of integration. If $\ell \geqq 1$, this singularity disappears by rewriting $(1/(g(g+\mu))^{1-\ell}$ as $(g(g+\mu))^{\ell-1}$. For $\ell < 1$, there are two alternatives for removing the singularity. We can transform the variable of integration, g, to become $t = g^\ell$ and the singularity at $g = 0$ is removed. Or, we could do an integration by parts to remove either the singularity at $g = 0$ for $u > 0$ or at $g = -u$ for $u < 0$. In either case, the remaining integral must be evaluated numerically for $u \neq 0$.

Since X is a symmetric random variable we can rewrite (III.B.1.9) using integration by parts to obtain an expression that will be easier to apply. For all $u \neq 0$

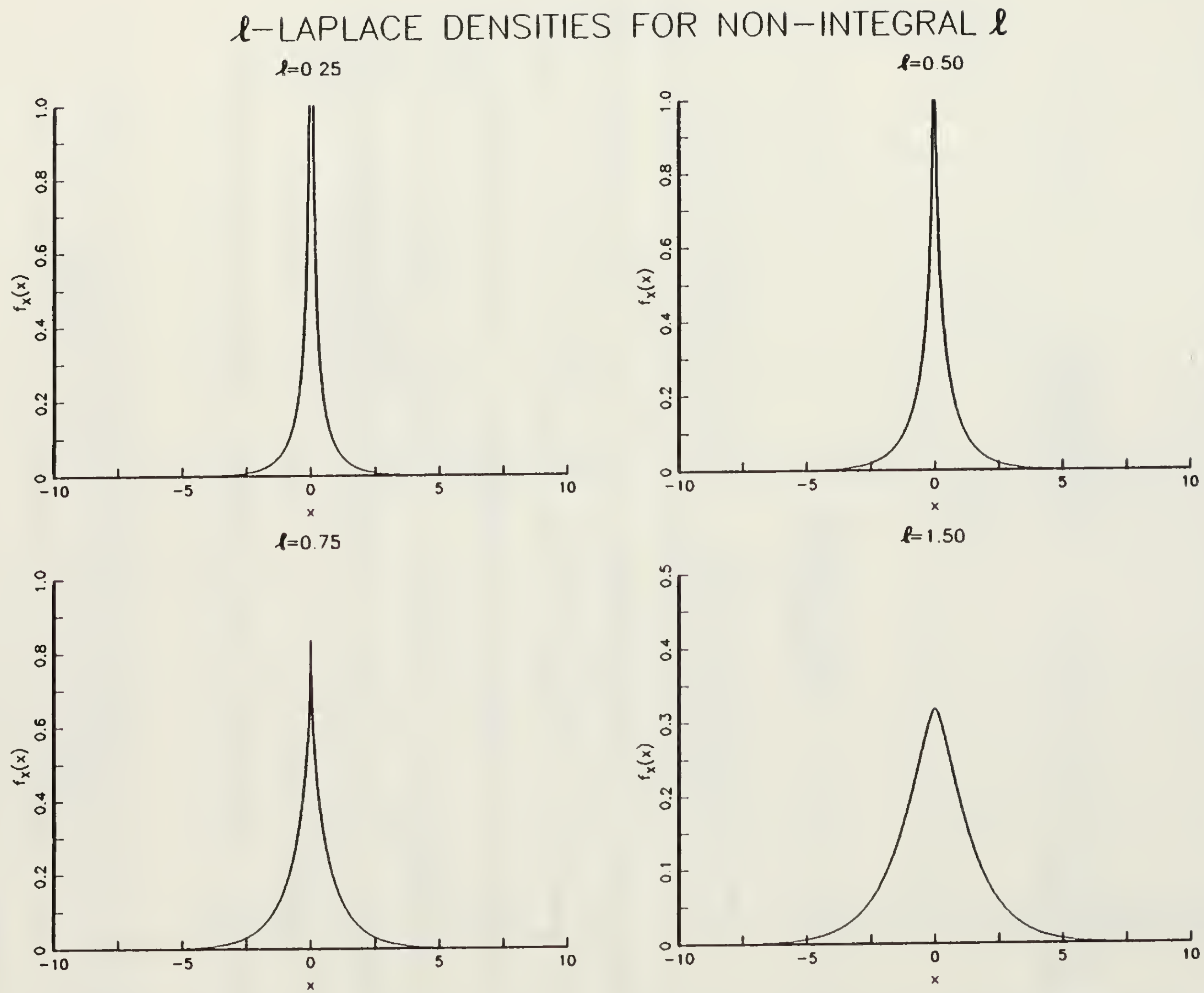


Figure III.B.2.1. Examples of the ℓ-Laplace Density for Non-Integral Values of ℓ Numerical Evaluation of Equation III.B.1.9

$$f_X(u;\ell) = \frac{\exp(-|u|)}{\ell\Gamma^2(\ell)} \int_{g=0}^{g=\infty} \frac{g^{\ell}\{2(g+|u|)+1-\ell\}}{(g+|u|)^{2-\ell}} \exp(-2g)dg. \qquad \text{(III.B.2.3)}$$

If $\ell \leq .5$ note that $f_X(u)$ is not defined at $\mu = 0$. For $\ell > .5$ and $u = 0$,

$$f_X(0;\ell) = \Gamma(2\ell-1)/\{\Gamma^2(\ell)\, 2^{2\ell-1}\} < \infty . \qquad \text{(III.B.2.4)}$$

3. <u>The Square Root Beta-Laplace Transformation</u>

The principal result of this section is the proof of the so-called square root Beta-Laplace transformation theorem. By this technique, an ℓ_1-Laplace random variable can be transformed into an ℓ_2-Laplace random variable where $\ell_2 \leq \ell_1$. The time series models developed in Sections III.C - III.F rely on the following:

<u>Theorem:</u>

Let $X \sim \ell$-Laplace and $B \sim \text{Beta}(\alpha,\ell-\alpha)$, where $0 < \alpha < \ell$ and B is defined on the interval [0,1], i.e. standard Beta. If $Y = B^{1/2}X$, then $Y \sim \alpha$-Laplace.

<u>Proof:</u>

By conditioning on B, we obtain the following expression for the characteristic function of Y;

$$\phi_Y(\omega) = E\{\exp(iB^{1/2}X\omega)\}$$

$$= E_B[E\{\exp(ib^{1/2}X\omega)\}]$$

$$= E_B[\{1/(1+b\omega^2)\}^{\ell}]. \qquad \text{(III.B.3.1)}$$

Since $b\omega^2 > 0$, a convergent power series representation of (III.B.3.1) is given by

$$\phi_Y(\omega) = E_B\{\sum_{k=0}^{\infty} \frac{\ell^{[k]}}{k!}(-\omega^2)^k b^k\}, \qquad \text{(III.B.3.2)}$$

where again $\ell^{[k]} = \ell(\ell+1)\ldots(\ell+k-1)$ for $k = 1,2,\ldots$; $\ell^{[0]} = 1$.

Interchanging the expectation and summation in a convergent power series gives

$$\phi_Y(\omega) = \sum_{k=0}^{\infty} \frac{\ell^{[k]}}{k!} (-\omega^2)^k \; E(B^k). \qquad \text{(III.B.3.3)}$$

From Johnson and Kotz [Ref. 36: v. 2, p. 40], we have

$$E(B^k) = \alpha^{[k]}/\ell^{[k]} \quad \text{for } k \text{ integer.} \qquad \text{(III.B.3.4)}$$

Substituting (III.B.3.4) and (III.B.3.3), we have

$$\phi_Y(\omega) = \sum_{k=0}^{\infty} \frac{\alpha^{[k]}}{k!}(-\omega^2)^k = (\frac{1}{1+\omega^2})^{\alpha}. \qquad \text{(III.B.3.5)}$$

Q.E.D.

C. ℓ-LAPLACE FIRST-ORDER AUTOREGRESSIVE TIME SERIES MODEL

1. Introduction

In this section, we exploit the square root Beta-Laplace transform to define a 2-parameter first-order autoregressive model in ℓ-Laplace variables. The first parameter, ℓ, determines the non-Gaussian symmetric marginal distribution of the time series ensemble. The second parameter, α, given the value of ℓ, determines uniquely the lag-1 serial correlation. Since the model is shown to be first-order Markovian, α determines the entire autocorrelation function up to the sign. We show also that the models are always partially time reversible with respect to both runs probabilities and directional moments.

Writing the stationary time series $\{X_n(\ell)\}$ in the form of an additive random coefficient equation, we have

$$X_n(\ell) = A_n^{1/2}(\alpha,\ell-\alpha)X_{n-1}(\ell) + B_n^{1/2}(\ell-\alpha,\alpha)L_n(\ell), \qquad \text{(III.C.1.1)}$$

where $\{X_n(\ell)\}$ is assumed to be a stationary time series with a marginal ℓ-Laplace distribution; $\{A_n^{1/2}(\alpha,\ell-\alpha)\}$ is an i.i.d. sequence such that $A_n(\alpha,\ell-\alpha)$ is a standard Beta; $\{B_n^{1/2}(\ell-\alpha,\alpha)\}$ is an i.i.d. sequence independent of $\{A_n^{1/2}(\alpha,\ell-\alpha)\}$ such that $B_n(\ell-\alpha,\alpha)$ is also standard Beta; and $\{L_n(\ell)\}$ is an i.i.d. sequence, independent of the coefficient processes, such that $L_n(\ell)$ is ℓ-Laplace. The coefficient $A_n(\alpha,\ell-\alpha)$ and $B_n(\ell-\alpha,\alpha)$ are assumed to be independent of X_{n-1}, X_{n-2}, etc. If it is assumed that $X_{n-1}(\ell)$ has a ℓ-Laplace distribution, then by the theorem in Section III.B.3 so does $X_n(\ell)$. The fact that the process is

Markovian follows by construction. To start the process in the stationary distribution set $X_0(\ell) = L_0(\ell)$.

The parameter space is $\ell > 0$ and $0 \leq \alpha \leq \ell$.

For the Beta random variables A_n and B_n (hence their square roots) to be properly defined, each of the parameters must be positive. Hence, when $\alpha = 0$ or $\alpha = \ell$, (III.C.1.1) as defined above is no longer appropriate because each of $A_n^{1/2}$ and $B_n^{1/2}$ has a parameter that is identically zero. If $\alpha = 0$, it is understood that the $\{A_n^{1/2}\}$ sequence is identically zero and the $\{B_n^{1/2}\}$ sequence is one; therefore, (III.C.1.1) becomes $X_n(\ell) = L_n(\ell)$ and the $\{X_n\}$ sequence is the $\{L_n\}$ corresponding to the i.i.d. ℓ-Laplace case. For $\alpha = \ell$, $A_n^{1/2}$ is one and $B_n^{1/2}$ is identically zero; therefore, if $X_0 = L_0(\ell)$, then X_n is ℓ-Laplace, but is not an ergodic process.

If we let

$$\varepsilon_n = B_n^{1/2}(\ell-\alpha,\alpha)L_n(\ell) \qquad \text{(III.C.1.2)}$$

then by the Theorem in Section III.B.3, $\varepsilon_n \sim (\ell-\alpha)$-Laplace with $E(\varepsilon_n) = 0$ and $Var(\varepsilon_n) = 2(\ell-\alpha)$ for all n. Since the variance must be non-negative, it is also necessary that $\alpha \leq \ell$. By the stationarity of $\{X_n\}$ and since $A_n^{1/2}(\alpha,\ell-\alpha)X_{n-1}(\ell)$ is independent of ε_n, the characteristic function of the right-hand side of (III.C.1.1) gives

$$\phi_{X(\ell)}(\omega) = \left\{\frac{1}{1+\omega^2}\right\}^{\alpha} \left\{\frac{1}{1+\omega^2}\right\}^{\ell-\alpha} = \left\{\frac{1}{1+\omega^2}\right\}^{\ell}. \qquad \text{(III.C.1.3)}$$

Examples of sample path behavior for selected ℓ and α are given in Figure III.C.1.1. Note that although the correlation coefficient is approximately 0.8 for all sets of ℓ and α in Figure III.C.1.1, there is considerable difference in the sample path behavior as ℓ changes. For the samples from small values of ℓ(.10 and .05), there are runs of values that are very nearly zero in magnitude.

2. Correlation Structure

Using equation (III.C.1.1) recursively along with the stationarity and independence of the process $\{X_n\}$, we have

$$\rho(1) = \mathrm{Corr}(X_n(\ell), X_{n-1}(\ell)) = \frac{E\{X_n(\ell)X_{n-1}(\ell)\}}{E\{X^2_{n-1}(\ell)\}}$$

$$= \frac{E[\{A_n^{1/2}(\alpha,\ell-\alpha)X_{n-1}(\ell)+\varepsilon_n\}X_{n-1}(\ell)]}{E\{X^2_{n-1}(\ell)\}}$$

$$= \frac{E\{A_n^{1/2}(\alpha,\ell-\alpha)\}E\{X^2_{n-1}(\ell)\}}{E\{X^2_{n-1}(\ell)\}} = E\{A_n^{1/2}(\alpha,\ell-\alpha)\}. \quad \text{(III.C.2.1)}$$

From Johnson and Kotz [Ref. 36: v. 2, p. 40], we have for $A_n \sim \mathrm{Beta}(\alpha,\ell-\alpha)$, that

$$E(A_n^r(\alpha,\ell-\alpha)) = \frac{\Gamma(\alpha+r)\Gamma(\ell)}{\Gamma(\ell+r)\Gamma(\alpha)} \quad \text{for all n, r > 0,} \quad \text{(III.C.2.2)}$$

where $\Gamma(.)$ is the incomplete Gamma function. Therefore

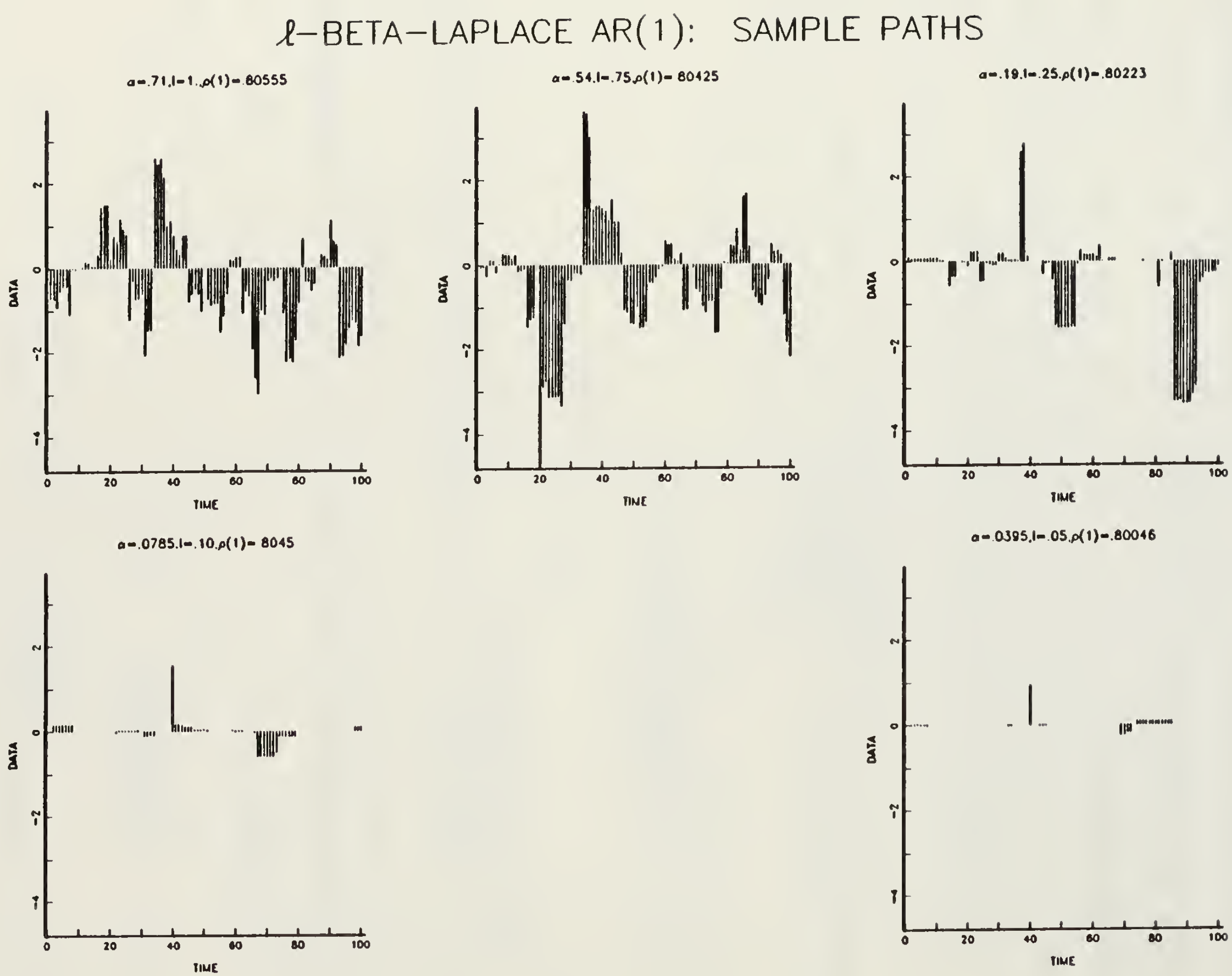


Figure III.C.1.1. ℓ-Beta-Laplace AR(1): Sample Paths; $\rho(1) \approx .8$

$$\rho(1) = \frac{\Gamma(\alpha+1/2)\Gamma(\ell)}{\Gamma(\ell+1/2)\Gamma(\alpha)} = \frac{\alpha\Gamma(\alpha+1/2)\Gamma(\ell+1)}{\ell\Gamma(\ell+1/2)\Gamma(\alpha+1)}. \qquad \text{(III.C.2.3)}$$

Note that as $\alpha \to \ell$, then $\rho(1) \to 1$. Similarly as $\alpha \to 0$, $\rho(1) \to 0$. Therefore, we obtain a full range of positive correlations in a one-to-one function of α for any given value of ℓ.

Also from (III.C.1.1), we see that the process is explicitly autoregressive. It is also autoregressive in the sense of expectations in that $E(X_n(\ell)|X_{n-1}(\ell) = x)$ is a linear function of x. Since (III.C.1.1) defines a first-order Markovian process, $\rho(k) = \rho(1)^{|k|}$ for all k. To see this we write for all k

$$\begin{aligned}
\rho(k) &= \frac{E\{X_n(\ell)X_{n-k}(\ell)\}}{2\ell} \\
&= E\{A_n^{1/2}(\alpha,\ell-\alpha)\}\frac{E\{X_{n-1}(\ell)X_{n-k}(\ell)\}}{2\ell} \\
&= \rho(1)\rho(k-1) \\
&= \rho(1)\rho(k-2)\rho(1) \\
&\vdots \\
&= \{\rho(1)\}^{|k|}.
\end{aligned}$$

If we replace $A_n^{1/2}(\alpha,\ell-\alpha)$ in (III.C.1.1) by $-A_n^{1/2}(\alpha,\ell-\alpha)$ we have

$$\rho(1) = -\frac{\Gamma(\alpha+1/2)\Gamma(\ell)}{\Gamma(\ell+1/2)\Gamma(\alpha)} = -\frac{\alpha\Gamma(\alpha+1/2)\Gamma(\ell+1)}{\ell\Gamma(\ell+1/2)\Gamma(\alpha+1)}. \qquad \text{(III.C.2.5)}$$

We can, therefore, achieve a full range of negative correlations, and likewise

$$\rho(k) = (-1)^{|k|}\{\rho(1)\}^{|k|} \quad \text{for all } k. \qquad \text{(III.C.2.6)}$$

3. Partial Time Reversibility

The ℓ-Laplace first-order autoregressive models are partially time reversible, both with respect to the directional moments, $\{X_n^2(\ell)X_{n-m}(\ell)\}$ for $m = 0, \pm 1, \pm 2, \ldots$, and with respect to runs probabilities, $P\{X_n(\ell)<X_{n-1}(\ell)\} = P\{X_n(\ell)>X_{n-1}(\ell)\}$.

Using mathematical induction, stationarity of $\{X_n(\ell)\}$, and the independence of the coefficients and the innovation from each other and previous values of $\{X_n(\ell)\}$, it is the case that $\{X_n^2(\ell)X_{n-m}(\ell)\}$ $= E\{X_n(\ell)X_{n-m}^2(\ell)\} = 0$ for all n and for all $m = 0, 1, 2, \ldots$. Let $X_n \sim \ell$-Laplace. For $m = 0$, $E(X_n^3) = 0$ by (III.B.1.2). Assuming for $m = k$ that $E(X_n^2X_{n-k}) = 0$, we have for $m = k+1$ after substituting from (III.C.1.1) and (III.C.1.2) that

$$\begin{aligned} E\{X_n^2X_{n-(k+1)}\} &= E\{(A_nX_{n-1}^2+2A_n^{1/2}X_{n-1}\varepsilon_n+\varepsilon_n^2)X_{n-(k+1)}\} \\ &= E(A_n)E\{X_{n-1}^2X_{n-(k+1)}\} \\ &= E(A_n)E(X_n^2X_{n-k}) = 0. \end{aligned} \qquad \text{(III.C.3.1)}$$

Assuming for $m = k$ that $E(X_nX_{n-k}^2) = 0$, we have for $m = k+1$ after substituting again from (III.C.1.1) and (III.C.1.2)

$$E\{X_n X^2_{n-(k+1)}\} = E\{X^2_{n-(k+1)}(A_n^{1/2} X_{n-1} + \varepsilon_n)\}$$

$$= E(A_n^{1/2}) E\{X^2_{n-(k+1)} X_{n-1}\}$$

$$= E(A_n^{1/2}) E(X^2_{n-k} X_n) = 0. \qquad \text{(III.C.3.2)}$$

To see that this model is also partially time reversible with respect to runs probabilities, we show that the random variable $Z_n = X_n - X_{n-1}$ is symmetric. Now Z_n is symmetric if and only if the characteristic function of Z_n is real valued. We write

$$\phi_Z(\omega) = E[\exp\{i\omega(X_n - X_{n-1})\}]$$

$$= E\left[\exp[i\omega\{\varepsilon_n - (1-A_n^{1/2})X_{n-1}\}]\right]$$

$$= E\{\exp(i\omega\varepsilon_n)\} E[\exp\{-i\omega(1-A_n^{1/2})X_{n-1}\}]$$

$$= \left\{\frac{1}{1+\omega^2}\right\}^{\ell-\alpha} E_A\left[E[\exp\{-i\omega(1-a^{1/2})X_{n-1}\}]\right]$$

$$= \left\{\frac{1}{1+\omega^2}\right\}^{\ell-\alpha} E_A\left\{\left\{\frac{1}{1+(1-a^{1/2})^2\omega^2}\right\}^{\ell}\right\}. \qquad \text{(III.C.3.3)}$$

Since (III.C.3.3) is real valued that concludes the proof.

D. THE BETA-LAPLACE AUTOREGRESSIVE MODEL, BELAR(1)

1. Introduction

In this section, we set $\ell = 1$ in (III.C.1.1) and (III.C.1.2) to obtain the following expression for the BELAR(1) process

$$X_n = A_n^{1/2}(\alpha, 1-\alpha)X_{n-1} + \varepsilon_n, \qquad \text{(III.D.1.1)}$$

where $\{\varepsilon_n\}$ is an i.i.d. sequence with $\varepsilon_n \sim (1-\alpha)$-Laplace with moments and density given by (III.B.1.2) and (III.B.2.3). X_n now has a standard Laplace marginal distribution. The only parameter in the model is α with $0 \leqq \alpha \leqq 1$. All the results of Section III.C still hold with $\ell = 1$. Examples of sample path behavior are given in Figure III.D.1.1.

We do two things in this section. First, we derive the equations for the conditional density of $X_n|X_{n-1}$. The second is the derivation of joint density and the logarithm of the likelihood function. The expression is used in Section III.E.6 to obtain the maximum likelihood estimate for α.

2. The Conditional Density

To find the conditional density of $X_n|X_{n-1}$, we will need the density of $A_n^{1/2}(\alpha, 1-\alpha)$. Let A_n be a standard Beta random variable with parameters $(\alpha, 1-\alpha)$. Since A_n is defined only on the given interval, zero to one

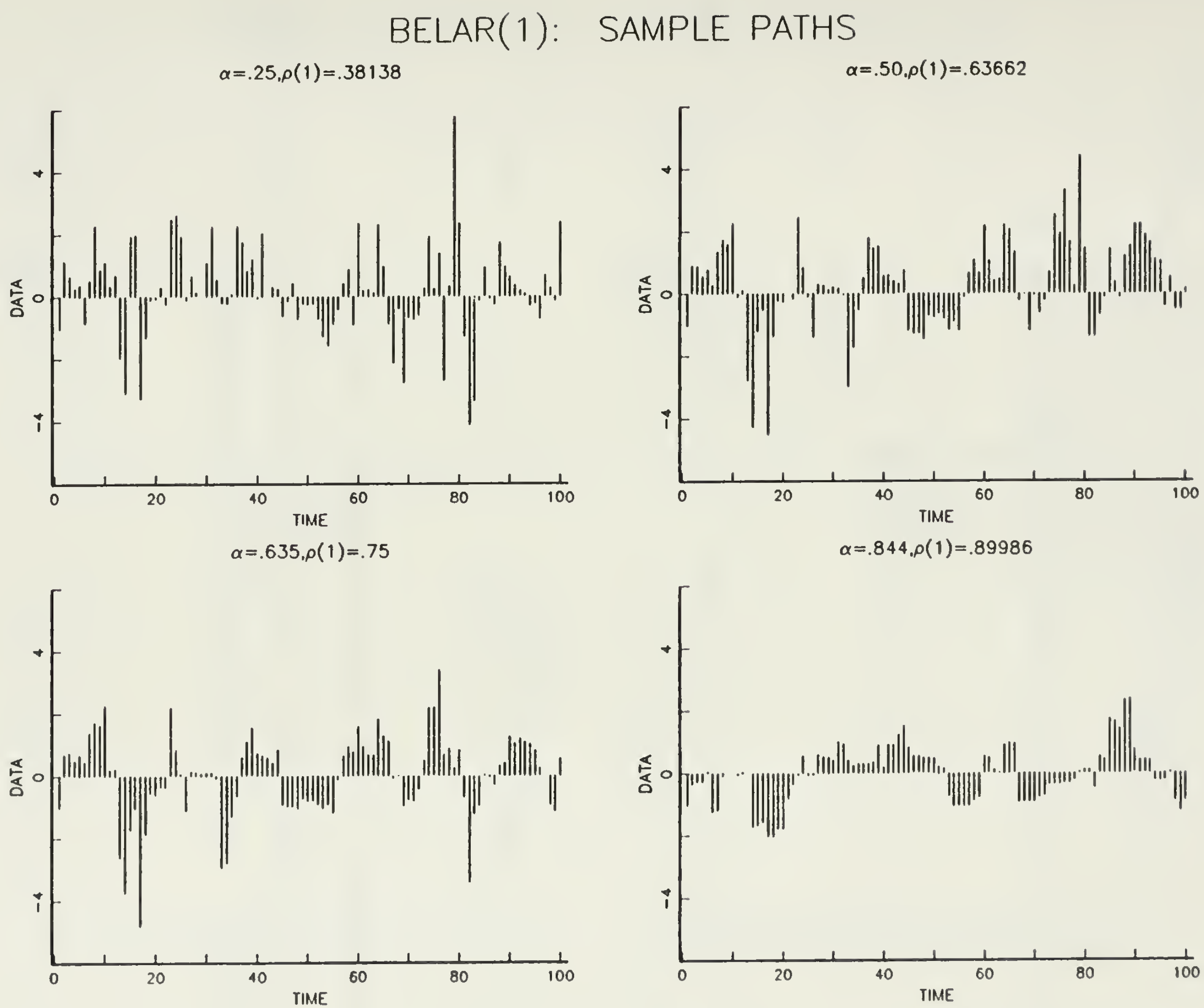


Figure III.D.1.1. BELAR(1): Sample Paths for Specified Values of α and Corresponding $\rho(1)=\gamma$

$$P(A_n^{1/2}<a) = \begin{cases} P(A_n<a^2) & 0 < a < 1 \\ 0 & \text{otherwise} \end{cases}$$

$$= \int_{x=0}^{x=a^2} f_{A_n}(x;\alpha)dx, \qquad 0 < a < 1, \qquad \text{(III.D.2.1)}$$

where $f_{A_n}(x;\alpha)$ is the standard Beta(α,1-α) density given by

$$f_{A_n}(x;\alpha) = \begin{cases} a^{\alpha-1}(1-a)^{\alpha}/\Gamma(\alpha)\Gamma(1-\alpha) & 0 < a < 1, \\ 0 & \text{otherwise.} \end{cases} \qquad \text{(III.D.2.2)}$$

Differentiating (III.D.2.1) with respect to a, we obtain the following expression for

$$f_{A_n^{1/2}}(a;\alpha) = \frac{2}{\Gamma(\alpha)\Gamma(1-\alpha)} \frac{a^{2\alpha-1}}{(1-a^2)^{\alpha}}, \qquad 0 < a < 1. \qquad \text{(III.D.2.3)}$$

Examples of (III.D.2.3) are given in Figure III.D.2.1.

Now we evaluate $P(X_n < x|X_{n-1}=y)$ using (III.D.1.1), (III.B.1.2), and (III.B.2.3). Conditioning on $A_n^{1/2}(\alpha,1-\alpha)$ we obtain

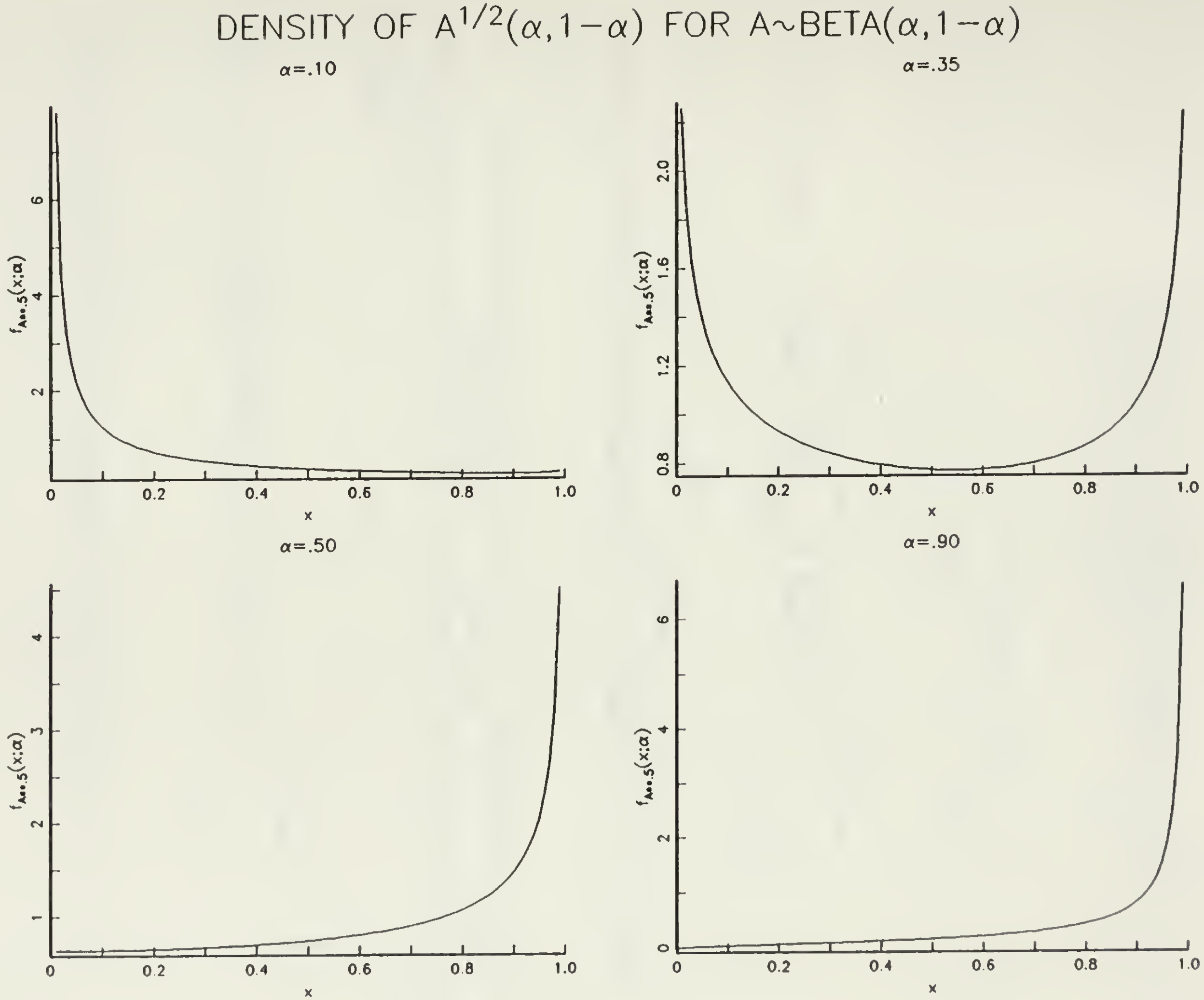


Figure III.D.2.1. Examples of the Density of $A_n^{1/2}(\alpha,1-\alpha)$ for Specified Values of $0<\alpha<1$

$$P(X_n<x|X_{n-1}=y) = P\{A_n^{1/2}(\alpha,1-\alpha)X_{n-1} + \varepsilon_n < x|X_{n-1}=y\}$$

$$= E_{A_n^{1/2}}[P\{\varepsilon_n < x - yA_n^{1/2}(\alpha,1-\alpha)|A_n^{1/2}=a\}]$$

$$= E_{A_n^{1/2}}\{P(\varepsilon_n < x-ay)\}$$

$$= E_{A_n^{1/2}}\{F_{\varepsilon_n}(x-ay)\}$$

$$= \int_{a=L_1(x)}^{a=L_2(x)} F_{\varepsilon_n}(x-ay)\, f_{A_n^{1/2}}(a;\alpha)\, da\ , \qquad \text{(III.D.2.4)}$$

where from (III.B.2.3) the cumulative distribution function of ε_n can be written as

$$F_{\varepsilon_n}(x-ay) = \begin{cases} \frac{1}{2} + \int_{u=0}^{u=x-ay} f_{\varepsilon_n}(u;1-\alpha)du & \text{if } x-ay \geq 0, \\ \\ \int_{u=0}^{u=ay-x} f_{\varepsilon_n}(u;1-\alpha)du & \text{if } x-ay < 0, \end{cases} \qquad \text{(III.D.2.5)}$$

and $L_i(x)$, $i = 1,2$ are the limits of integration on a which may be functions of x.

Since $F_{\varepsilon_n}(x-ay)$ changes definition for negative and positive $(x-ay)$ and since $0 < a < 1$, we rewrite (III.D.2.4) based on the ratio x/y, which is a constant. Thus

$$P(X_n<x|X_{n-1}=y) = \begin{cases} \displaystyle\int_{a=0}^{a=1} F_{\varepsilon_n}(x-ay)f_{A_n^{1/2}}(a;\alpha)da & \text{if } x/y \geq 1 \text{ or } x/y \leq 0; \\ \\ \displaystyle\int_{a=0}^{a=x/y} F_{\varepsilon_n}(x-ay)f_{A_n^{1/2}}(a;\alpha)da & \\ \quad\displaystyle + \int_{a=x/y}^{a=1} F_{\varepsilon_n}(x-ay)f_{A_n^{1/2}}(a;\alpha)da & \text{if } 0 < x/y < 1 . \end{cases} \qquad \text{(III.D.2.6)}$$

Differentiating (III.D.2.4) with respect to x using Leibniz' rule gives the following general expression for the conditional density. We have

$$f_{X_n|X_{n-1}}(x|y) = \frac{d}{dx}\{P(X_n<x|X_{n-1}=y)\}$$

$$= \int_{a=L_1(x)}^{a=L_2(x)} f_{\varepsilon_n}(x-ay;1-\alpha)f_{A_n^{1/2}}(a;\alpha)da$$

$$+ F_{\varepsilon_n}\{x-yL_2(x)\}f_{A_n^{1/2}}\{L_2(x);\alpha\} \frac{d}{dx}L_2(x)$$

$$- F_{\varepsilon_n}\{x-yL_1(x)f_{A_n^{1/2}}\{L_1(x);\alpha\}\frac{d}{dx}L_1(x). \qquad \text{(III.D.2.7)}$$

From (III.B.2.3), (III.D.2.3) and (III.D.2.5) set

$$h(g,a) = \frac{2a^{2\alpha-1}\exp\{-(2g+|x-ay|)\}\; g^{1-\alpha}(2g+2|x-ay|+\alpha)}{\Gamma^3(1-\alpha)\;\Gamma(\alpha)\;(1-\alpha^2)^{\alpha}\;(g+|x-ay|)^{1+\alpha}\;(1-\alpha)}\;. \qquad \text{(III.D.2.8)}$$

Now using (III.D.2.7) to differentiate each expression in (III.D.2.6), we have the following explicit expressions for

$$f_{X_n|X_{n-1}}(x|y) = \begin{cases} \displaystyle\int_{a=0}^{a=1}\int_{g=0}^{g=\infty} h(g,a)dgda & \text{if } \; x/y \geqq 1 \; \text{ or } \; x/y \leqq 0\,, \\ \\ \displaystyle\int_{a=0}^{a=x/y}\int_{g=0}^{g=\infty} h(g,a)dgda & \\ \displaystyle\qquad + \int_{a=x/y}^{a=1}\int_{g=0}^{g=\infty} h(g,a)dgda & \text{if } \; 0 < x/y < 1\,. \end{cases} \qquad \text{(III.D.2.9)}$$

It will be seen later that working with (III.D.2.9) will be inconvenient. Hence, we rewrite (III.D.2.9) as

$$f_{X_n|X_{n-1}}(x|y) = \begin{cases} \displaystyle\int_{a=0}^{a=1} f_{\varepsilon_n}\{(x-ay);1-\alpha\} f_{A_n^{1/2}}(a;\alpha)da & \text{if } x/y \geq 1 \text{ or } x/y \leq 0 \\ \displaystyle\int_{a=0}^{a=x/y} f_{\varepsilon_n}\{(x-ay);1-\alpha\} f_{A_n^{1/2}}(a;\alpha)da \\ \quad + \displaystyle\int_{a=x/y}^{a=1} f_{\varepsilon_n}\{(x-ay);1-\alpha\} f_{A_n^{1/2}}(a;\alpha)da & \text{if } 0 < x/y < 1. \end{cases} \qquad \text{(III.D.2.10)}$$

The conditional density in (III.D.2.10) can assume different shapes as a function of x depending on the fixed conditioning value, y, and the particular, fixed α. If $\alpha = 0$, then (III.D.2.10) becomes the standard Laplace density as given in (II.B.1.1) with $\mu = 0$ and $\lambda = 1$. If $y = 0$, then (III.D.2.10) becomes the $(1-\alpha)$-Laplace density as given in (III.B.2.3) with $\ell = 1-\alpha$. In Figure III.D.2.2 are presented different examples of (III.D.2.10) for a fixed y and different values of α. Note that if $\alpha < 1/2$ then (III.D.2.10) is continuous for all x. If $\alpha \geq 1/2$ and $x = y$, (III.D.2.10) is undefined, e.g., $x = y = 0$.

In a similar manner, expressions for (III.D.2.4)-(III.D.2.10) can be derived for the BELAR(1) model with negative correlations. Placing $-A_n^{1/2}(\alpha,1-\alpha)$ in (III.D.1.1), we replace x-ay by x+ay and determine the appropriate form of the conditional density based on the ratio (-x/y). We have for the negative BELAR(1) process

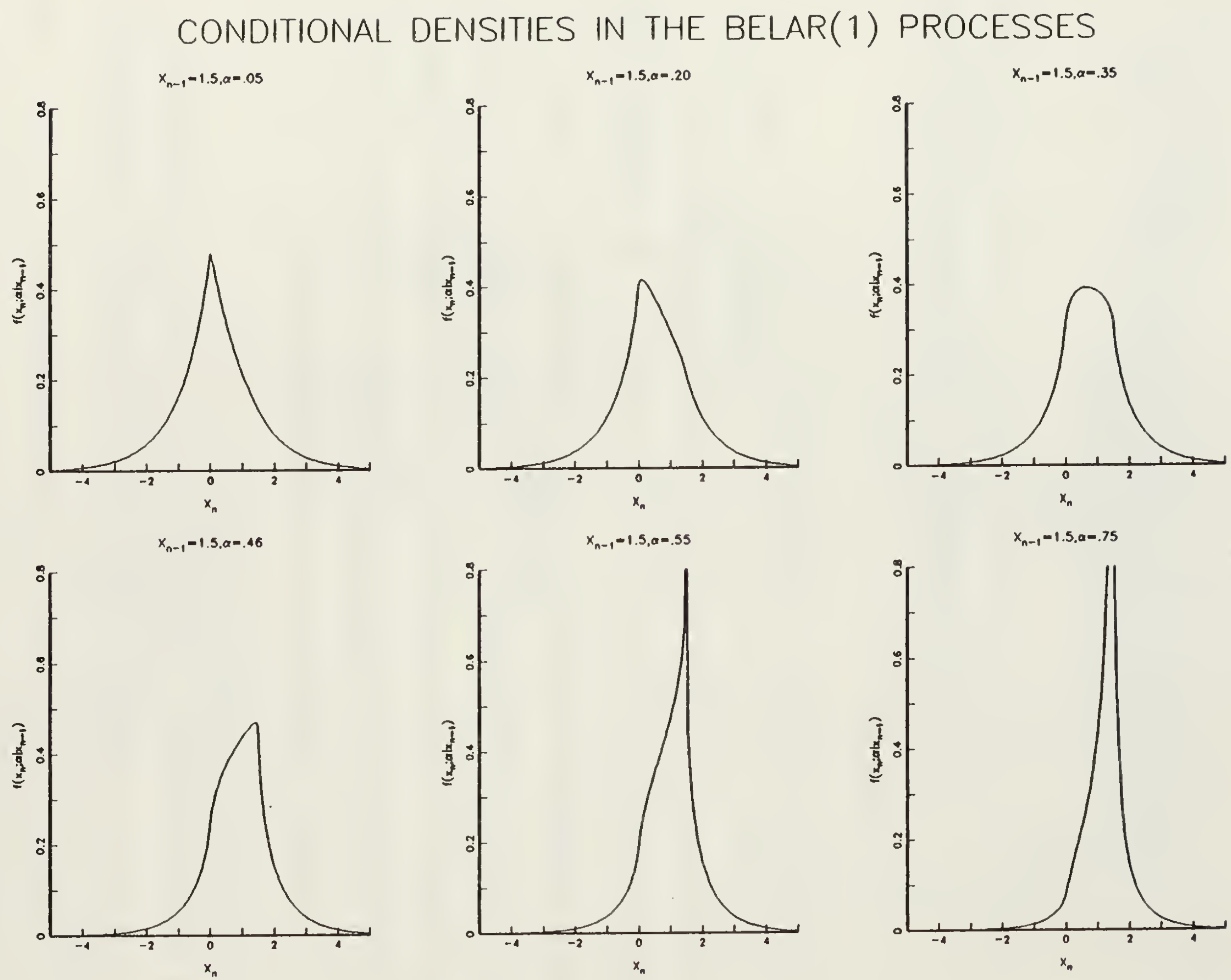


Figure III.D.2.2. Examples of Conditional Density of X_n Given X_{n-1} in the BELAR(1) Process

$$f_{X_n|X_{n-1}}(x|y) = \begin{cases} \displaystyle\int_{a=0}^{a=1} f_{\varepsilon_n}\{(x+ay);1-\alpha\} f_{A_n^{1/2}}(a;\alpha)\,da \\ \qquad \text{if } -x/y \geq 1 \text{ or } -x/y \geq 0, \\ \\ \displaystyle\int_{a=0}^{a=-x/y} f_{\varepsilon_n}\{(x+ay),1-\alpha\} f_{A_n^{1/2}}(a;\alpha)\,da \\ \quad + \displaystyle\int_{a=-x/y}^{a=1} f_{\varepsilon_n}\{(x+ay);1-\alpha\} f_{A_n^{1/2}}(a;\alpha)\,da \\ \qquad \text{if } 0 < -x/y < 1. \end{cases} \qquad \text{(III.D.2.11)}$$

3. The Joint Distribution and the Likelihood Function

An expression for the joint density of $X_n,\ldots,X_1$ can be written using $f_{X_n|X_{n-1}}(x_n|x_{n-1})$ and $f_{X_1}(x_1)$ as follows:

$$f_{X_n\ldots X_1}(x_n,\ldots,x_1) = f_{X_1}(x_1) \prod_{k=1}^{n-1} f_{X_{n-(k-1)}|X_{n-k}}(x_{n-(k-1)}|x_{n-k}). \qquad \text{(III.D.3.1)}$$

The log-likelihood function as a function of α given $\{X_n\}$ is just the natural logarithm of (III.D.3.1). We have

$$L(\alpha) = -(\ln 2 + |x_1|) + \sum_{k=1}^{n-1} \ln\{f_{X_{n-(k-1)}|X_{n-k}}(x_{n-(k-1)}|x_{n-k})\}. \qquad \text{(III.D.3.2)}$$

It is now a simple matter to determine which branch of (III.D.2.10) or (III.D.2.11) is needed for each pair (x_n, x_{n-1}) and to substitute it into the sum in (III.D.3.2). We postpone further discussion of the likelihood function until Section III.E.6.

4. Numerical Evaluation of the Conditional Density

a. Introduction

This section is devoted to explaining the methodology by which we came to resolve the problems in the numerical integration of the conditional density. This is as important an issue as the derivation itself, since the likelihood function and the maximum likelihood estimators can not be evaluated without it. As is pointed out below, the standard numerical routines were unsuccessful in accurately evaluating (III.D.2.9) around the singularities in (III.D.2.8). We also give and justify the approximations that were used to remove each of the singularities. The graphs in Figure III.D.2.2 were obtained using the method. The methodology was used again in Section III.E.6 to evaluate the log-likelihood function in the method of maximum likelihood estimation.

In the FORTRAN routine that calculates the conditional density as given in (III.D.2.10), the approximations in (III.D.4.6), (III.D.4.8) and (III.D.4.11) are added to the results from DCADRE. Combinations of these approximations are invoked as necessary depending on the ratio x/y.

The same procedure is used to evaluate the density in (III.D.2.11) for the BELAR(1) model which produces negative correlations

for odd lags. We just check for $0 < -x/y < 1$ and choose the appropriate value of c in (III.D.4.6) and (III.D.4.8) where x-ay is replaced by x+ay.

b. The Methodology

Attempts to evalute the conditional density, as given by (III.D.2.8) and (III.D.2.9), using the standard IMSL double integration routines failed. Even the IMSL routine DBLIN which is often successful in handling ill-behaved integrands, was unable to evaluate (III.D.2.8) around the singularities. For $\alpha < 1/2$, along the lines $a = 0$ and $a = 1$, (III.D.2.8) is unbounded. Similarly for $\alpha \geqq 1/2$, along the line $a = 1$ and at the point $(g,a) = (0,x/y)$ for $0 < x/y < 1$, (III.D.2.8) is unbounded. Arbitrarily declaring (III.D.2.8) to be zero under these conditions did not always allow DBLIN to accurately evaluate (III.D.2.9).

We succeeded in evaluating the conditional density by working with the form given by (III.D.2.10) with $f_{A_n^{1/2}}(a;\alpha)$ given by (III.D.2.3) and $f_{\varepsilon_n}\{(x-ay);1-\alpha\}$ given by (III.B.2.3). We used the IMSL routine DCADRE to construct an extensive table of values for the $(1-\alpha)$-Laplace density with the intention to linearly interpolate from the table as needed. The error in the value of $f_\varepsilon(|u|;1-\alpha)$ in the table is controlled by DCADRE. The error in the value of $f_\varepsilon(|u_0|;1-\alpha)$ obtained by using linear interpolation for $|u_0|$ not in the table is calculated in the standard way. From Gerald [Ref. 28: p. 168]

$$|\text{Error Interpolation}| = \left| \frac{h^2 s(s-1)}{2} \quad \frac{d^2 f_{\varepsilon_n}(c;1-\alpha)}{du^2} \right| , \qquad \text{(III.D.4.1)}$$

where h is subinterval length and $s = (u_0-u)/h$. Substituting the second divided difference into (III.D.4.1), in place of the unknown second derivative and also noting that the worst case for linear interpolation is at the center of the subinterval, we have

$$|\text{Error Interpolation}| < \left| -\frac{1}{8} \, \Delta^2 f_{\varepsilon_n}(|u|;1-\alpha) \right| , \qquad \text{(III.D.4.2)}$$

where $\Delta^2 f_{\varepsilon_n}$ is the second difference. Because $f_{\varepsilon_n}(|u|;1-\alpha)$ is non-negative and monotone decreasing in $|u|$, the largest values of $\Delta^2 f$ are in subintervals close to zero. The table that was constructed, therefore, uses smaller subintervals close to zero and larger subintervals further out.

Finally we used DCADRE again to evaluate (III.D.2.10) except near the singularities, which we were able to evaluate analytically and then add back. The technique is often referred to as "removing the singularity".

c. Removing the Singularities Due to (III.D.2.3)

We now describe how we evaluated the integrals in (III.D.2.10) in the vicinity of the singularities in (III.D.2.3). We see that the density of $A_n^{1/2}(\alpha,1-\alpha)$ given in (III.D.2.3) is undefined at $a = 0$ and $a = 1$ for $\alpha < 1/2$ and at $a = 1$ for $\alpha \geqq 1/2$. We also note from (III.D.2.3) that for small $\delta > 0$ and $\alpha < 1/2$

$$f_{A_n^{1/2}}(a;\alpha) \doteq \frac{2a^{2\alpha-1}}{\Gamma(\alpha)\Gamma(1-\alpha)}, \quad 0 < a < \delta; \qquad \text{(III.D.4.3)}$$

and for all $0 < \alpha < 1$

$$f_{A_n^{1/2}}(a;\alpha) \doteq \frac{2a}{\Gamma(\alpha)\Gamma(1-\alpha)(1-a^2)^{\alpha}}, \quad 1-\delta < a < 1 . \qquad \text{(III.D.4.4)}$$

Therefore for $\alpha < 1/2$ and $1 \leqq x/y$ or $x/y \leqq 0$ we have from (III.D.4.3)

$$\int_{a=0}^{a=\delta} f_{A_n^{1/2}}(a;\alpha) f_{\varepsilon_n}\{(x-ay);1-\alpha\}da \doteq \int_{a=0}^{a=\delta} \frac{2a^{2\alpha-1}}{\Gamma(\alpha)\Gamma(1-\alpha)} f_{\varepsilon_n}\{(x-ay);1-\alpha\}da. \qquad \text{(III.D.4.5)}$$

Since $f_{\varepsilon_n}(\cdot)$ is continuous in this situation, there exists a number c so that $0 < c < \delta$ and $|x| \leqq |x-cy| \leqq |x-\delta y|$ and

$$\int_{a=0}^{a=\delta} \frac{2a^{2\alpha-1}}{\Gamma(\alpha)\Gamma(1-\alpha)} f_{\varepsilon_n}\{(x-ay);1-\alpha\}da = f_{\varepsilon_n}\{(x-cy);1-\alpha\} \int_{a=0}^{a=\delta} \frac{2a^{2\alpha-1}}{\Gamma(\alpha)\Gamma(1-\alpha)}\, da$$

$$= f_{\varepsilon_n}\{(x-cy);1-\alpha\} \frac{(1-\alpha)\delta^2}{\Gamma(2-\alpha)\Gamma(1+\alpha)} . \qquad \text{(III.D.4.6)}$$

A natural approximation for c allows $|x-cy|$ to be the average, $(1/2)|2x-\delta y|$.

For all α and $1 < x/y$ or $x/y < 0$ we have from (III.D.4.4)

$$\int_{a=1-\delta}^{a=1} f_{A_n^{1/2}}(a;\alpha) f_{\varepsilon_n}\{(x-ay);1-\alpha\}da$$

$$\approx \int_{a=1-\delta}^{a=1} \frac{1}{\Gamma(\alpha)\Gamma(1-\alpha)} \frac{2a}{(1-a^2)^\alpha} f_{\varepsilon_n}\{(x-ay);1-\alpha\}da \ . \qquad \text{(III.D.4.7)}$$

Likewise there exists a new number c so that $1-\delta < c < 1$ and $|x-y| < |x-cy| < |x-y+\delta y|$ and

$$\int_{a=1-\delta}^{a=1} \frac{1}{\Gamma(\alpha)\Gamma(1-\alpha)} \frac{2a}{(1-a^2)^\alpha} f_{\varepsilon_n}\{(x-ay);1-\alpha\}da$$

$$= f_{\varepsilon_n}\{(x-cy);1-\alpha\} \int_{a=1-\delta}^{a=1} \frac{1}{\Gamma(\alpha)\Gamma(1-\alpha)} \left(\frac{2a}{(1-a^2)^\alpha}\right)da$$

$$= f_{\varepsilon_n}\{(x-cy);1-\alpha\} \frac{\alpha(2\delta)^{1-\alpha}}{\Gamma(1+\alpha)\Gamma(2-\alpha)} \ . \qquad \text{(III.D.4.8)}$$

Again a natural approximation for c allows $|x-cy|$ to be the average, $(1/2)|2(x-y)+\delta y|$.

d. Removing the Singularity Due to (III.B.2.3)

The final type of singularity occurs when $0 < a = x/y < 1$ and $\alpha \geqq 1/2$. When this situation occurs we leave $f_{\varepsilon_n}(\cdot)$ under the integral and argue that in a δ-neighborhood around $x/y < 1$, $f_{A_n^{1/2}}(a;\alpha) \approx f_{A_n^{1/2}}(x/y;\alpha)$. Note that by the same argument that gave us

(III.D.4.6) and (III.D.4.8), there exist two numbers c_1 and c_2 so that $(x/y)-\delta \leq c_1 \leq x/y$ and $x/y \leq c_2 \leq (x/y)+\delta$ and

$$\int_{a=(x/y)-\delta}^{a=x/y} f_{A_n^{1/2}}(a;\alpha) f_{\varepsilon_n}\{(x-ay);1-\alpha\}da$$

$$+ \int_{a=x/y}^{a=(x/y)+\delta} f_{A_n^{1/2}}(a;\alpha) f_{\varepsilon_n}\{(x-ay);1-\alpha\}da$$

$$= f_{A_n^{1/2}}(c_1;\alpha) \int_{a=(x/y)-\delta}^{a=x/y} f_{\varepsilon_n}\{(x-ay);1-\alpha\}da$$

$$+ f_{A_n^{1/2}}(c_2;\alpha) \int_{a=x/y}^{a=(x/y)+\delta} f_{\varepsilon_n}\{(x-ay);1-\alpha\}da. \qquad \text{(III.D.4.9)}$$

We chose to approximate c_1 and c_2 both by x/y for $x/y \neq \pm 1$, and have $f_{A_n^{1/2}}(x/y;\alpha) < \infty$ for all α. If $x/y = 1$ or $x = 0$ and $y = 0$ simultaneously, the value of (III.D.2.10) is undefined for $\alpha \geqq 1/2$.

Now changing the variable of integration so that $(x-ay) = u$, we have from (III.D.4.9) that for all $\alpha \geqq 1/2$

$$\int_{a=(x/y)-\delta}^{a=x/y} f_{\varepsilon_n}\{(x-ay);1-\alpha\}da = \int_{a=x/y}^{a=(x/y)+\delta} f_{\varepsilon_n}\{(x-ay);\alpha\}da,$$

$$= \frac{1}{|y|} \int_{u=0}^{u=|y\delta|} f_{\varepsilon_n}(u;1-\alpha)du$$

$$\leqq \left(\frac{1}{2}\right) \frac{1}{|y|} , \qquad \text{(III.D.4.10)}$$

because $f_{\varepsilon_n}(\cdot)$ is a symmetric density. That is (III.D.4.10) is an expression for $\frac{1}{|y|} P(0 < \varepsilon_n < |y\delta|)$ where ε_n is the $(1-\alpha)$-Laplace innovation random variable. Therefore, we add back to the DCADRE result the amount

$$\left(\frac{1}{|y|}\right) f_{A_n^{1/2}}(x/y;\alpha)\{P(0 < \varepsilon_n < |y\delta|)\} \leqq \left(\frac{1/2}{|y|}\right) f_{A_n^{1/2}}(x/y;\alpha) < \infty , \ y \neq 0. \qquad \text{(III.D.4.11)}$$

We choose the following combination as the value for $P(0 < \varepsilon_n < |y\delta|)$.

i) Using the trapezoidal rule and the table of values for the $(1-\alpha)$-Laplace density we found

$$P_1(0 < \varepsilon_n < |y\delta|) = 1/2 - \int_{u=|y\delta|}^{u=M} f_{\varepsilon_n}(u;1-\alpha)du . \qquad \text{(III.D.4.12)}$$

Equation (III.D.4.12) is the average of the upper and lower Riemann sums of the tail of the density subtracted from 1/2. Using (III.D.4.12) instead of directly integrating $f_{\varepsilon_n}(u;1-\alpha)$ from zero to $|y\delta|$ is preferrable, because for $\alpha \geqq 1/2$, $f_{\varepsilon_n}(0;1-\alpha)$ is undefined. The error in

(III.D.4.12) from using the trapezoidal rule approximation is approximately $\left|\frac{h_i^3}{12}\,\Delta^2 f_{\varepsilon_n}(i)\right|$ in the i^{th} subinterval. Even though there are over 400 subintervals, the second differences $\Delta^2 f_{\varepsilon}(i)$ are very much smaller for $\alpha \geq 1/2$ in the interval $[|y\delta|, M]$.

ii) A second measure of $P(0<\varepsilon_n<|y\delta|)$ is the lower sum

$$P_2(0 < \varepsilon_n < |y\delta|) = |y\delta| f_{\varepsilon_n}\{(y\delta);1-\alpha\}, \qquad \text{(III.D.4.13)}$$

since $P(0 < \varepsilon_n < |y\delta|)$ is always at least as large as (III.D.4.13). Our approximation for $P(0 < \varepsilon_n < |y\delta|)$ is the maximum of (III.D.4.12) and (III.D.4.13). We use the maximum because P_1 given by (III.D.4.12) could be negative when $|y\delta|$ is close to zero. This follows because $F_{\varepsilon_n}(u;1-\alpha)$ is strictly decreasing for $u > 0$, and thus the trapezoidal rule overestimates the integral in (III.D.4.12).

E. PARAMETER ESTIMATION IN THE BELAR(1) PROCESS

1. Introduction

In this section, we develop estimators for the parameters in the BELAR(1) process and report results on properties of these estimators obtained from analytical comparisons and simulations. We examine estimators for the location parameter, μ, and the scale paramter, λ, of the series $\{X_n\}$; the parameter, α, of the random coefficient $A_n^{1/2}(\alpha,1-\alpha)$; and Υ, the lag-1 serial correlation, which is a monotone function of α.

The theory of conditional least squares estimation for the BELAR(1) process using the linearized residual is derived using results from Nicholls and Quinn [Ref. 16]. We give a corollary to their Theorem 3.1 pertaining to the strong convergence and asymptotic Normality of the least squares estimator of γ, the lag-1 serial correlation. An estimate for α is derived using the fact that $\gamma = E\{A_n^{1/2}(\alpha, 1-\alpha)\}$. Also, we show that the joint least squares estimator of location and correlation for the BELAR(1) process is the same as for the linear AR(1) processes.

Other estimators of lag-1 serial correlation in the BELAR(1) process are derived using the ideas of robust estimation of Huber [Ref. 37] and least absolute deviation (LAD) estimation as applied to ordinary linear autoregressive models by Denby and Martin [Ref. 38] and Bloomfield and Steiger [Ref. 39]. Although these estimators are consistent and asymptotically unbiased in linear models, for the random coefficient models the results of the simulation study show that they have a bias that does not go to zero asymptotically.

The maximum likelihood estimator of α, $\hat{\alpha}_{MLE}$, is found using an iterative technique with the initial estimate being derived from the least squares estimate of serial correlation, $\hat{\gamma}_{LS}$.

Many of the simulations comparing the different estimators are conducted within the framework of SIMTBED [Ref. 15]. From the Summary Statistics table generated by SIMTBED for each estimator, it is possible to draw conclusions concerning the bias, the variance at different subsample sizes, the asymptotic variance, and how fast the estimator approaches asymptotic Normality. In the SIMTBED program, one can specify the total number of samples examined at each subsample size.

The total number of samples used is the product of three parameters, N, M, and NSR. Three combinations of these parameters were used. Table III.E.1.1 is a summary of the number and types of subsample sizes, N, and the number of independent repetitions, M, of each type of simulation conducted using SIMTBED.

TABLE III.E.1.1

Summary of SIMTBED Types

	Subsample Sizes (N)								Number of Super Replications
Type	25	50	75	100	125	175	250	500	(NSR)
I	2000	1000	660	500	400	280	200	100	5
II	4000	2000	1330	1000	800	570	400	200	10
III	8000	4000	2660	2000	1600	1140	800	400	10

Each entry in a Summary Statistics table, which is the output of SIMTBED after super replication, is a pair corresponding to a mean (average over the number of super replications, 5 or 10) and an estimated standard deviation of that mean value. From Table III.E.1.1, it is clear that a large number of independent realizations was used in the computation for each super replication and the different subsample sizes. Because of this, subsequent tests of hypothesis that we use on the simulation outputs will be t-tests on the mean of a random sample of size 5 or 10 drawn from a Normal population where σ^2 is unknown, but is estimated from the sample.

Before describing each estimator and simulation experiment, it is convenient now to summarize the conclusions of this investigation into the estimation of parameters in the BELAR(1) process:

a. The simulation results from SIMTBED indicate that both the sample median and sample mean are asymptotically Normal estimators of μ. The asymptotic variance of the sample mean is approximately twice that of the sample median across all values of the correlation coefficient, γ.

b. The simulation results from SIMTBED also indicate that the mean absolute deviation, given in (II.E.3.2), is an unbiased and asymptotically Normal estimator of the scale parameter, λ. It also has the smallest asymptotic variance of the three estimators considered.

c. The least squares estimator of γ, the lag-1 serial correlation is asymptotically unbiased and Normally distributed. Simulation results support this conclusion.

d. Simulation of other estimators of lag-1 serial correlation based on non-linear residuals of the form $\tilde{R}_n = X_n - \gamma X_{n-1} + \beta f(X_n, X_{n-1})$ indicates that the value of (γ, β) that maximizes the sum of squares of $\tilde{R}_n$ is approximately $(\hat{\gamma}_{LS}, 0)$.

e. Robust estimators of serial correlation based on certain symmetric loss functions of the linear residual (other than the sum of squares) are biased and, apparently, asymptotically biased. SIMTBED outputs of the Huber(c), rank and LAD estimators of lag-1 serial correlation clearly exhibited this result.

f. The maximum likelihood estimator of γ, the lag-1 serial correlation was computed by the iteration scheme given in Section III.E.6 for simulated data from the BELAR(1) process. Results of the simulation appear to indicate that the estimator is converging

to a Normal distribution with a mean value equal to the true γ. In comparison to the least squares estimator, the simulation results indicate that the maximum likelihood estimator has a smaller variance and bias at all values of γ.

2. Estimators of Location

a. Introduction

The sample median, m, and the sample mean, $\overline{X}$, are two commonly used estimators of the location parameter, μ, in a stationary process with a symmetric marginal distribution. The sample median is a particularly attractive alernative to $\overline{X}$ when the symmetric distribution is also thick-tailed. (It is well known that for i.i.d. processes with a double exponential marginal distribution that the sample median is the maximum likelihood estimator of μ).

For i.i.d. processes, it is well known (Dudewiez, [Ref. 40: p. 221]) that $\overline{X}$ has an asymptotically Normal distribution, $N(0,\sqrt{\sigma_X^2/n})$. Likewise, m is asymptotically Normal, $N\{0, \sqrt{1/4nf_X^2(x_{.5})}\}$. The results for the sample median hold provided $f_X(x_{.5})$ is continuous in a neighborhood around $x_{.5}$, is positive, and is bounded above.

The problem of estimating μ from dependent data is more difficult. Analytical results exist about the limiting distribution for $\overline{X}$ in ergodic processes and for the sample median for processes satisfying certain mixing conditions. (Mixing processes are those for which random variables "sufficiently far apart" are approximately independent).

Since the BELAR(1) process is an RCA(1) process with i.i.d. innovation and random coefficient processes, $\{X_n\}$, is ergodic (Nicholls and Quinn [Ref. 16: p. 37]). Therefore $\bar{X}$ is still an unbiased asymptotically Normal estimator of μ, but the variance is modified by the factor

$$1 + 2 \sum_{k=1}^{\infty} \gamma^k = (1+\gamma)/(1-\gamma). \qquad \text{(III.E.2.1)}$$

See, for example, Priestly [Ref. 33: p. 343].

The problem of estimating the median has been studied for cases where the data are dependent. From Heidelberger and Lewis [Ref. 41)], we have that the usual order statistic point estimate (sample median) is still valid, but the variance is modified by a factor, $p(x_{.5})$. Here $p(x_{.5})$ is the initial point on the spectrum of the binary process $\{I_n(x_{.5})\}$, where

$$I_n(x) = \begin{cases} 1 & \text{if } X_n \leqq x, \\ 0 & \text{otherwise.} \end{cases} \qquad \text{(III.E.2.2)}$$

That is

$$p(x_{.5}) = \lim_{n\to\infty} n \operatorname{Var} \{ \sum_{i=1}^{n} I_i(x_{.5})/n \}. \qquad \text{(III.E.2.3)}$$

As was already pointed out, conditions for convergence and Central Limit Theorems for the sample median depend on mixing

conditions. There are several kinds of mixing conditions. It is not known, however, if the BELAR(1) process satisfies any of them.

However, the LAR(1) process does satisfy the mixing conditions of Gastwirth and Rubin [Ref. 14]. Thus, for the LAR(1) process, it is known that the sample median has an asymptotic Normal distribution with mean zero, and variance given by

$$\sum_{k=-\infty}^{+\infty} \{\gamma^{|k|}\cosh(x_{.5}\gamma^{|k|}) + \sinh(x_{.5}\gamma^{|k|})\} = \left(\frac{1+\gamma}{1-\gamma}\right). \qquad \text{(III.E.2.4)}$$

Gastwirth and Rubin [Ref. 14] showed that for the LAR(1) process, the asymptotic variance of $\overline{X}$ is twice that of the sample median across all values of serial correlation.

The question here is, what are the properties of the sample median in estimating μ from data of the BELAR(1) process? Also, how does the sample median compare to $\overline{X}$ in the BELAR(1) process?

Since $\{X_n\}$ from both the BELAR(1) and the LAR(1) processes have a marginal Laplace distribution and first-order autoregressive correlation structure, the hypothesis is that the sample median from the BELAR(1) process behaves similarily to that generated from data in the LAR(1) process. Also, the relative efficiency of m to $\overline{X}$ is the same in the two processes.

To substantiate this assumption, the sample median and sample mean were compared in simulation experiments in SIMTBED for data generated from the BELAR(1) process. The simulation output is compared to the theoretical results for the LAR(1) process.

b. Simulation Results

For $\alpha = .1$ and a corresponding correlation coefficient of $\gamma = .17664$, the estimators $\overline{X}$ and m were simulated in SIMTBED using a size of Type III from Table III.E.1.1. The results are given in the Summary Statistics in Table III.E.2.1. Looking at Table III.E.2.1 for N = 100 and greater, there is no evidence of non-Normality from the first four estimated moments of the sample mean. The leading coefficient in the asymptotic expansions for $E(\overline{X})$ and $Var(\overline{X})$ do not deviate significantly from the theoretical values, i.e. $\overline{X}$ is unbiased and $Var(\overline{X}) \approx 2.8581/N$.

Looking at Table III.E.2.2, the Summary Statistics at $\alpha = .1$ for m, it appears that even for N = 25, m is unbiased and the sample skewness is fluctuating about zero. The variance, however, at each subsample size up to N = 250 deviates significantly from a hypothetical asymptotic variance of 1.4291/N, the corresponding result for LAR(1). This is explained by the kurtosis of the estimate m of the median which, although decreasing with increased subsample size, is still significantly different from 0 until N = 250. The leading coefficients in the expansions for the expectation and for the variance are not significantly different from 0 and 1.4291 respectively. Since the data are only slightly correlated, we could have expected the sample median to behave similarily to that of the case of the completely random process with Laplace marginals, i.e. m is unbiased, asymptotically Normal, and has a variance with leading coefficient 1/n.

TABLE III.E.2.1

SIMTBED Summary Statistics for Estimating μ by $\overline{X}$ in the BELAR(1) Process with α=.1 and γ=.17664

SUMMARY STATISTICS (MEAN/STD) 10 REPETITIONS

SUBSAMPLE SIZE	25	50	75	100	125	175	250	500
MEAN	-0.9024E-04 0.1774E-02	0.4031E-03 0.1433E-02	-0.1606E-02 0.3502E-03	-0.1773E-02 0.1131E-02	-0.6640E-03 0.1272E-02	0.3226E-03 0.9407E-03	-0.2113E-04 0.1099E-02	0.1271E-02 0.1459E-02
STD	0.3340 0.1055E-02	0.2360 0.8081E-03	0.1947 0.9639E-03	0.1688 0.4158E-03	0.1515 0.7664E-03	0.1267 0.7621E-03	0.1058 0.1098E-02	0.7698E-01 0.1008E-02
SKEWNESS	0.1689E-01 0.8302E-02	0.2376E-01 0.1359E-01	0.1909E-01 0.1732E-01	0.1036E-01 0.1518E-01	0.3714E-02 0.2827E-01	0.9780E-01 0.1682E-01	-0.6775E-01 0.5516E-01	-0.2833E-02 0.1500E-01
KURTOSIS	0.2541 0.1816E-01	0.1449 0.3246E-01	0.1003 0.3951E-01	0.7352E-01 0.4504E-01	0.2610E-01 0.4749E-01	-0.1113E-02 0.5157E-01	0.3666E-01 0.7278E-01	-0.1206 0.7011E-01
SER.COR.	-0.6153E-02 0.3754E-02	-0.7255E-02 0.3199E-02	-0.3032E-02 0.5615E-02	-0.3826E-02 0.1001E-01	0.4173E-02 0.8968E-02	-0.6308E-02 0.1248E-01	0.1246E-02 0.1133E-01	-0.5244E-02 0.1410E-01
QUANTILES								
0.010	-0.7940 0.5396E-02	-0.5510 0.5200E-02	-0.4569 0.4805E-02	-0.3974 0.4818E-02	-0.3547 0.6651E-02	-0.2922 0.5291E-02	-0.2494 0.3489E-02	-0.1753 0.4913E-02
0.025	-0.6559 0.4436E-02	-0.4635 0.3476E-02	-0.3815 0.3005E-02	-0.3339 0.3129E-02	-0.3005 0.4811E-02	-0.2459 0.3274E-02	-0.2090 0.4105E-02	-0.1503 0.4644E-02
0.050	-0.5479 0.2547E-02	-0.3865 0.2602E-02	-0.3214 0.2107E-02	-0.2766 0.2399E-02	-0.2498 0.2425E-02	-0.2081 0.2040E-02	-0.1762 0.3187E-02	-0.1255 0.2241E-02
0.100	-0.4236 0.2477E-02	-0.3004 0.1853E-02	-0.2493 0.1291E-02	-0.2155 0.2501E-02	-0.1941 0.2309E-02	-0.1602 0.2097E-02	-0.1353 0.2214E-02	-0.9691E-01 0.1810E-02
0.250	-0.2206 0.1708E-02	-0.1566 0.1418E-02	-0.1330 0.1391E-02	-0.1155 0.1576E-02	-0.1026 0.1562E-02	-0.8570E-01 0.1446E-02	-0.7101E-01 0.1929E-02	-0.5247E-01 0.1677E-02
0.500	-0.1080E-02 0.1565E-02	0.1130E-02 0.2230E-02	-0.2376E-02 0.6506E-03	-0.2669E-02 0.8272E-03	-0.4011E-03 0.1530E-02	-0.7146E-03 0.7996E-03	0.1535E-03 0.1509E-02	0.6831E-03 0.1641E-02
0.750	0.2189 0.2275E-02	0.1572 0.1917E-02	0.1281 0.9980E-03	0.1117 0.1423E-02	0.1008 0.2424E-02	0.8425E-01 0.1243E-02	0.7307E-01 0.2062E-02	0.5396E-01 0.1961E-02
0.900	0.4233 0.2696E-02	0.2987 0.1820E-02	0.2474 0.1992E-02	0.2123 0.2736E-02	0.1932 0.2017E-02	0.1650 0.1738E-02	0.1342 0.2403E-02	0.1012 0.2345E-02
0.950	0.5489 0.3860E-02	0.3883 0.2727E-02	0.3193 0.2015E-02	0.2793 0.2291E-02	0.2477 0.1644E-02	0.2092 0.1923E-02	0.1735 0.2915E-02	0.1307 0.2520E-02
0.975	0.6629 0.3117E-02	0.4703 0.2690E-02	0.3809 0.1527E-02	0.3363 0.3273E-02	0.2968 0.3462E-02	0.2503 0.2457E-02	0.2027 0.2516E-02	0.1529 0.2202E-02
0.990	0.7970 0.4564E-02	0.5647 0.4295E-02	0.4542 0.4404E-02	0.3982 0.5617E-02	0.3579 0.4557E-02	0.3037 0.5451E-02	0.2461 0.5525E-02	0.1857 0.3741E-02

MEAN OF REGRESSION ON AVERAGES - COEFFICIENTS:	0.305551E-02 0.177728E-02	-0.828425 0.440196	54.9052 30.9138	-860.071 535.899
STD DEV OF REGRESSION - COEFFICIENTS:	0.527513E-02 0.311186E-03	1.39691 0.771368E-01	93.5843 5.58091	1586.93 95.1910
REGRESSION ON VARIANCE - COEFFICIENTS:	2.85720 0.248420	1.18737 6.00316	-29.2873 47.5942	63.2432 120.908

ESTIMATOR: SAMPLE MEAN; MU=0.0

*** WIDEST Y VALUES FOUND: YMIN=-1.629 YMAX= 1.797

TABLE III.E.2.2

SIMTBED Summary Statistics for Estimating μ by m in the BELAR(1) Process with α=.1 and γ=.17664

SUMMARY STATISTICS (MEAN/STD)   10 REPETITIONS

SUBSAMPLE SIZE	25	50	75	100	125	175	250	500
MEAN	-0.8201E-03 0.1185E-02	-0.4663E-04 0.9243E-03	-0.9629E-03 0.6889E-03	-0.7370E-03 0.6880E-03	0.5218E-03 0.1207E-02	-0.3256E-03 0.1051E-02	-0.4603E-03 0.6322E-03	0.1356E-02 0.9865E-03
STD	0.2677 0.7503E-03	0.1795 0.7238E-03	0.1449 0.7625E-03	0.1244 0.6775E-03	0.1093 0.6920E-03	0.9244E-01 0.6541E-03	0.7584E-01 0.6597E-03	0.5431E-01 0.7418E-03
SKEWNESS	-0.3412E-02 0.8548E-02	0.1775E-02 0.1276E-01	0.2479E-01 0.1777E-01	-0.6894E-02 0.2260E-01	-0.2134E-02 0.2837E-01	0.2261E-01 0.3093E-01	0.4103E-01 0.3960E-01	0.1981E-01 0.4382E-01
KURTOSIS	0.8428 0.5117E-01	0.5053 0.3687E-01	0.4053 0.4064E-01	0.3702 0.4108E-01	0.2784 0.5237E-01	0.2419 0.5594E-01	0.2560 0.1103	0.1393E-01 0.7670E-01
SER.COR.	-0.5105E-02 0.3868E-02	-0.5864E-02 0.5764E-02	-0.2479E-02 0.7819E-02	0.4713E-02 0.5340E-02	-0.9390E-02 0.5619E-02	-0.8915E-02 0.1305E-01	0.6203E-02 0.9719E-02	-0.1286E-01 0.1483E-01
QUANTILES								
0.010	-0.6690 0.4034E-02	-0.4355 0.2860E-02	-0.3506 0.4411E-02	-0.3036 0.3844E-02	-0.2595 0.3837E-02	-0.2191 0.2720E-02	-0.1793 0.4753E-02	-0.1259 0.4712E-02
0.025	-0.5441 0.2624E-02	-0.3627 0.2313E-02	-0.2889 0.1832E-02	-0.2506 0.2345E-02	-0.2202 0.2748E-02	-0.1840 0.1852E-02	-0.1496 0.1774E-02	-0.1064 0.2598E-02
0.050	-0.4417 0.2176E-02	-0.2968 0.2002E-02	-0.2387 0.1266E-02	-0.2056 0.2375E-02	-0.1806 0.2314E-02	-0.1522 0.1405E-02	-0.1248 0.2068E-02	-0.8867E-01 0.1791E-02
0.100	-0.3326 0.2135E-02	-0.2256 0.1307E-02	-0.1837 0.1411E-02	-0.1582 0.1320E-02	-0.1382 0.1277E-02	-0.1177 0.1454E-02	-0.9687E-01 0.1146E-02	-0.6862E-01 0.2081E-02
0.250	-0.1636 0.1238E-02	-0.1134 0.8620E-03	-0.9337E-01 0.8605E-03	-0.8024E-01 0.1459E-02	-0.7082E-01 0.1719E-02	-0.6161E-01 0.1119E-02	-0.5034E-01 0.1087E-02	-0.3485E-01 0.1380E-02
0.500	0.3041E-03 0.1033E-02	0.7394E-03 0.9751E-03	-0.1496E-02 0.6753E-03	-0.6276E-03 0.9843E-03	0.7026E-03 0.1430E-02	-0.6528E-03 0.1203E-02	-0.1508E-02 0.6603E-03	0.1879E-02 0.9417E-03
0.750	0.1625 0.1213E-02	0.1134 0.1039E-02	0.9046E-01 0.1229E-02	0.7937E-01 0.1426E-02	0.7156E-01 0.1878E-02	0.5990E-01 0.1349E-02	0.4930E-01 0.1303E-02	0.3681E-01 0.1100E-02
0.900	0.3280 0.1873E-02	0.2238 0.2412E-02	0.1841 0.1826E-02	0.1535 0.2017E-02	0.1401 0.1805E-02	0.1169 0.2342E-02	0.9598E-01 0.1509E-02	0.7196E-01 0.1852E-02
0.950	0.4375 0.3121E-02	0.2958 0.2468E-02	0.2409 0.3230E-02	0.2032 0.2658E-02	0.1811 0.1842E-02	0.1530 0.2388E-02	0.1256 0.2128E-02	0.9201E-01 0.2116E-02
0.975	0.5426 0.3312E-02	0.3605 0.2105E-02	0.2915 0.3641E-02	0.2467 0.2451E-02	0.2182 0.1851E-02	0.1860 0.2910E-02	0.1540 0.2245E-02	0.1107 0.2805E-02
0.990	0.6744 0.3370E-02	0.4335 0.5037E-02	0.3501 0.3405E-02	0.3030 0.2880E-02	0.2611 0.2983E-02	0.2237 0.4256E-02	0.1857 0.2694E-02	0.1285 0.2566E-02

MEAN OF REGRESSION ON AVERAGES - COEFFICIENTS:	0.179135E-02 0.148104E-02	-0.473540 0.349875	27.6280 22.1594	-435.531 366.965
STD DEV OF REGRESSION - COEFFICIENTS:	0.407440E-02 0.311681E-03	1.07669 0.771763E-01	71.5511 5.54417	1202.51 96.0980
REGRESSION ON VARIANCE - COEFFICIENTS:	1.36657 0.177602	1.82473 4.73201	-4.69940 38.9729	30.9969 99.6975

ESTIMATOR: SAMPLE MEDIAN; MU=0.0
*** WIDEST Y VALUES FOUND: YMIN=-1.754   YMAX= 1.521

For values of α = .5 and .844, with corresponding γ = .63662 and .89986, using Type II experiments as described in Table III.E.1.1, we again compared the behavior of $\overline{X}$ and m.

From Tables III.E.2.3 and III.E.2.4, we see that the behavior of $\overline{X}$ is as expected. The sample mean appears to be unbiased. For N ≧ 250, there is no evidence of non-Normality. The estimates of the leading coefficient in the asymptotic expansions for the variance agree within one standard deviation of the postulated values of 9.0 and 38.

The corresponding results for m are given in Tables III.E.2.5 and III.E.2.6. The sample median shows no bias and appears to be asymptotically Normal after N ≧ 250. In each case (α = .5 and α = .844) the leading coefficient in the expansion for the variance is smaller than the corresponding value for the variance of the sample median in the LAR(1) process, i.e. 4.5 and 19 respectively.

The analysis thus far has indicated that at least for data with non-negative correlation in the BELAR(1) process, there is little evidence to suggest that the behavior of the sample median is significantly different than in the LAR(1) process. From Table III.E.2.7, we see the same kind of results that Gastwirth and Rubin [Ref. 14] reported. As sample size increases, the efficiency of $\overline{X}$ relative to m drops to 50%.

TABLE III.E.2.3

SIMTBED Summary Statistics for Estimating μ by $\overline{X}$ in the BELAR(1) Process with α=.5 and γ=.63662

SUMMARY STATISTICS (MEAN/STD) 10 REPETITIONS

SUBSAMPLE SIZE	25	50	75	100	125	175	250	500
MEAN	0.1099E-02 0.2465E-02	0.4443E-03 0.2898E-02	-0.3534E-03 0.3959E-02	-0.2034E-02 0.2933E-02	0.3287E-02 0.3197E-02	0.4625E-03 0.2533E-02	0.5943E-03 0.3787E-02	-0.2876E-02 0.3025E-02
STD	0.5757 0.2336E-02	0.4113 0.2438E-02	0.3439 0.2127E-02	0.3001 0.2386E-02	0.2659 0.2073E-02	0.2273 0.1023E-02	0.1910 0.2651E-02	0.1349 0.1957E-02
SKEWNESS	-0.6570E-01 0.2836E-01	0.1547E-01 0.2084E-01	0.7113E-02 0.2145E-01	0.3529E-01 0.2732E-01	0.2260E-01 0.2284E-01	-0.2081E-01 0.3504E-01	0.1389E-01 0.3499E-01	-0.3513E-01 0.3699E-01
KURTOSIS	0.8976 0.9168E-01	0.4419 0.6947E-01	0.2923 0.9702E-01	0.2879 0.6929E-01	0.6097E-01 0.5219E-01	0.2472 0.6084E-01	0.1326E-01 0.7556E-01	-0.8678E-01 0.4964E-01
SER.COR.	-0.2143E-02 0.5382E-02	-0.1591E-01 0.4911E-02	-0.9068E-02 0.9136E-02	-0.2269E-02 0.9257E-02	0.6232E-02 0.1420E-01	-0.3885E-01 0.9963E-02	0.1913E-01 0.1863E-01	0.2556E-01 0.1934E-01
QUANTILES								
0.010	-1.429 0.1447E-01	-0.9947 0.1310E-01	-0.8136 0.1234E-01	-0.7195 0.1273E-01	-0.6090 0.8306E-02	-0.5499 0.1150E-01	-0.4460 0.1337E-01	-0.3284 0.1348E-01
0.025	-1.152 0.1035E-01	-0.8217 0.7703E-02	-0.6807 0.5888E-02	-0.5908 0.8820E-02	-0.5075 0.5884E-02	-0.4522 0.9663E-02	-0.3758 0.1158E-01	-0.2703 0.9254E-02
0.050	-0.9296 0.7600E-02	-0.6690 0.5734E-02	-0.5611 0.6076E-02	-0.4920 0.8445E-02	-0.4246 0.5526E-02	-0.3692 0.5444E-02	-0.3152 0.8593E-02	-0.2289 0.8296E-02
0.100	-0.7068 0.3940E-02	-0.5101 0.5295E-02	-0.4366 0.5096E-02	-0.3810 0.5410E-02	-0.3399 0.6363E-02	-0.2891 0.3920E-02	-0.2444 0.4523E-02	-0.1765 0.4038E-02
0.250	-0.3589 0.4258E-02	-0.2685 0.2543E-02	-0.2313 0.3838E-02	-0.2003 0.3615E-02	-0.1772 0.4158E-02	-0.1519 0.1997E-02	-0.1293 0.5830E-02	-0.9238E-01 0.4683E-02
0.500	0.3580E-02 0.3461E-02	-0.6030E-03 0.2966E-02	0.2603E-02 0.5506E-02	-0.1720E-02 0.3941E-02	-0.4454E-03 0.3963E-02	0.2101E-02 0.4477E-02	0.2996E-02 0.4183E-02	-0.1097E-02 0.3715E-02
0.750	0.3643 0.4139E-02	0.2653 0.3013E-02	0.2233 0.6386E-02	0.1936 0.3181E-02	0.1803 0.3691E-02	0.1535 0.4432E-02	0.1297 0.3989E-02	0.9052E-01 0.4475E-02
0.900	0.7105 0.3729E-02	0.5146 0.5156E-02	0.4337 0.8070E-02	0.3778 0.4469E-02	0.3456 0.6278E-02	0.2850 0.2969E-02	0.2443 0.3722E-02	0.1694 0.3893E-02
0.950	0.9356 0.5563E-02	0.6788 0.6179E-02	0.5648 0.6332E-02	0.4986 0.6654E-02	0.4385 0.6020E-02	0.3754 0.5585E-02	0.3155 0.6470E-02	0.2167 0.4763E-02
0.975	1.148 0.9714E-02	0.8291 0.7668E-02	0.6731 0.8251E-02	0.6001 0.1030E-01	0.5364 0.8042E-02	0.4459 0.9401E-02	0.3801 0.8385E-02	0.2567 0.6094E-02
0.990	1.402 0.9002E-02	1.014 0.1464E-01	0.8202 0.1244E-01	0.7314 0.1339E-01	0.6477 0.1210E-01	0.5429 0.8807E-02	0.4617 0.1046E-01	0.3122 0.9700E-02

MEAN OF REGRESSION ON AVERAGES - COEFFICIENTS:	-0.274603E-02 0.596574E-02	0.714916 1.57120	-44.4773 98.1844	738.372 1580.82
STD DEV OF REGRESSION - COEFFICIENTS:	0.123439E-01 0.839364E-03	3.38546 0.272710	832.046 19.3303	3944.23 320.644
REGRESSION ON VARIANCE - COEFFICIENTS:	7.79841 1.28255	42.7098 34.9743	-425.731 299.260	1121.91 788.937

ESTIMATOR: SAMPLE MEAN; MU=0.0

TABLE III.E.2.4

SIMTBED Summary Statistics for Estimating μ by $\overline{X}$ in the BELAR(1) Process with α=.844 and γ=.89986

SUMMARY STATISTICS (MEAN/STD) 10 REPETITIONS

SUBSAMPLE SIZE	25	50	75	100	125	175	250	500
MEAN	0.2973E-02 0.3349E-02	-0.6042E-02 0.4511E-02	-0.3947E-02 0.4392E-02	0.3564E-02 0.8617E-02	-0.1372E-02 0.5862E-02	-0.2186E-02 0.5668E-02	-0.5689E-02 0.8305E-02	0.2247E-02 0.9895E-02
STD	0.9950 0.5199E-02	0.7953 0.5024E-02	0.6660 0.6101E-02	0.5906 0.4803E-02	0.5327 0.4589E-02	0.4524 0.4314E-02	0.3870 0.5635E-02	0.2688 0.3982E-02
SKEWNESS	-0.1916E-01 0.3783E-01	-0.7375E-02 0.4507E-01	0.2391E-01 0.5965E-01	0.8627E-01 0.5822E-01	-0.4870E-01 0.2916E-01	-0.9011E-01 0.4178E-01	-0.4067E-01 0.5571E-01	0.4682E-02 0.7997E-01
KURTOSIS	2.150 0.1644	1.482 0.1011	1.327 0.1365	0.8042 0.9451E-01	0.5972 0.1099	0.5795 0.8627E-01	0.3368 0.1130	0.7961E-01 0.1362
SER.COR.	0.2509E-02 0.4659E-02	0.8737E-02 0.7029E-02	-0.1348E-01 0.7709E-02	-0.7412E-02 0.1153E-01	-0.8927E-02 0.1335E-01	0.1020E-01 0.1629E-01	-0.8040E-02 0.1704E-01	-0.4481E-02 0.1943E-01
QUANTILES								
0.010	-2.649 0.1645E-01	-2.082 0.3178E-01	-1.686 0.2671E-01	-1.444 0.4023E-01	-1.334 0.2747E-01	-1.140 0.3118E-01	-0.9575 0.3280E-01	-0.6686 0.3269E-01
0.025	-2.060 0.2000E-01	-1.634 0.2125E-01	-1.347 0.2330E-01	-1.168 0.2843E-01	-1.062 0.1206E-01	-0.9294 0.1983E-01	-0.7921 0.1345E-01	-0.5386 0.2271E-01
0.050	-1.606 0.1278E-01	-1.303 0.6927E-02	-1.087 0.2016E-01	-0.9537 0.1669E-01	-0.8715 0.1183E-01	-0.7585 0.2096E-01	-0.6503 0.1484E-01	-0.4367 0.1282E-01
0.100	-1.161 0.6415E-02	-0.9620 0.7875E-02	-0.8117 0.1382E-01	-0.7236 0.1379E-01	-0.6694 0.8672E-02	-0.5700 0.9529E-02	-0.5023 0.1152E-01	-0.3309 0.1791E-01
0.250	-0.5481 0.4127E-02	-0.4747 0.4947E-02	-0.4119 0.6062E-02	-0.3602 0.6274E-02	-0.3399 0.8060E-02	-0.2835 0.6228E-02	-0.2563 0.9970E-02	-0.1818 0.1077E-01
0.500	0.5085E-02 0.3069E-02	-0.3983E-02 0.5099E-02	-0.8960E-02 0.5056E-02	-0.8395E-02 0.9025E-02	-0.3879E-02 0.7574E-02	0.5904E-02 0.7161E-02	0.1319E-03 0.8772E-02	-0.4978E-02 0.9830E-02
0.750	0.5582 0.6149E-02	0.4688 0.6245E-02	0.4011 0.6776E-02	0.3708 0.8801E-02	0.3391 0.1112E-01	0.2872 0.5907E-02	0.2423 0.7781E-02	0.1873 0.1106E-01
0.900	1.165 0.7463E-02	0.9383 0.7858E-02	0.8113 0.9661E-02	0.7355 0.1064E-01	0.6589 0.1143E-01	0.5517 0.9035E-02	0.4706 0.1621E-01	0.3545 0.1041E-01
0.950	1.596 0.1431E-01	1.273 0.2018E-01	1.085 0.1349E-01	0.9943 0.1310E-01	0.8851 0.9290E-02	0.7333 0.1411E-01	0.6219 0.1749E-01	0.4490 0.1545E-01
0.975	2.041 0.2477E-01	1.599 0.2083E-01	1.335 0.2155E-01	1.215 0.2191E-01	1.082 0.1648E-01	0.8982 0.1593E-01	0.7622 0.1962E-01	0.5460 0.1821E-01
0.990	2.646 0.3235E-01	2.058 0.2883E-01	1.680 0.3939E-01	1.502 0.4603E-01	1.311 0.2472E-01	1.121 0.1786E-01	0.9075 0.2573E-01	0.6537 0.1923E-01

MEAN OF REGRESSION ON AVERAGES - COEFFICIENTS:	-0.226423E-02 0.115725E-01	0.598353 2.55920	-64.5786 155.753	1321.64 2479.02
STD DEV OF REGRESSION - COEFFICIENTS:	0.266667E-01 0.168967E-02	7.28211 0.578575	474.398 39.0532	7780.32 640.677
REGRESSION ON VARIANCE - COEFFICIENTS:	36.8928 4.42441	24.8131 120.067	-491.458 982.394	320.733 2481.14

ESTIMATOR: SAMPLE MEAN; MU=0.0

TABLE III.E.2.5

SIMTBED Summary Statistics for Estimating μ by m in the BELAR(1) Process with α=.5 and γ=.63662

SUMMARY STATISTICS (MEAN/STD) 10 REPETITIONS

SUBSAMPLE SIZE	25	50	75	100	125	175	250	500
MEAN	0.1931E-02 0.1681E-02	-0.6091E-04 0.2109E-02	-0.2239E-03 0.2820E-02	0.4286E-03 0.1638E-02	0.1234E-02 0.2847E-02	0.5567E-03 0.1760E-02	0.1436E-02 0.2458E-02	-0.6970E-03 0.2026E-02
STD	0.4777 0.1884E-02	0.3119 0.2588E-02	0.2547 0.2035E-02	0.2159 0.1810E-02	0.1881 0.1158E-02	0.1583 0.8036E-03	0.1308 0.1625E-02	0.9139E-01 0.1616E-02
SKEWNESS	-0.6276E-01 0.4462E-01	0.6296E-01 0.3540E-01	0.9518E-02 0.2634E-01	0.3444E-01 0.3649E-01	0.4732E-01 0.2810E-01	-0.6669E-01 0.4025E-01	0.4632E-01 0.3817E-01	-0.6537E-02 0.4657E-01
KURTOSIS	2.161 0.1455	1.149 0.1688	0.8200 0.9454E-01	0.7672 0.1166	0.4491 0.9103E-01	0.5313 0.1102	0.3702 0.9604E-01	0.1669 0.7793E-01
SER.COR.	-0.1281E-02 0.5608E-02	-0.8517E-02 0.5499E-02	-0.2865E-02 0.1104E-01	0.3213E-02 0.9129E-02	0.2838E-02 0.1015E-01	-0.1417E-01 0.1231E-01	-0.4061E-02 0.1607E-01	0.3535E-01 0.2493E-01
QUANTILES								
0.010	-1.247 0.1025E-01	-0.7963 0.1320E-01	-0.6282 0.1174E-01	-0.5435 0.1127E-01	-0.4507 0.6166E-02	-0.3967 0.7414E-02	-0.3113 0.5248E-02	-0.2204 0.5682E-02
0.025	-0.9775 0.7219E-02	-0.6294 0.7975E-02	-0.5195 0.4742E-02	-0.4369 0.4120E-02	-0.3700 0.5295E-02	-0.3227 0.8635E-02	-0.2596 0.5805E-02	-0.1859 0.5627E-02
0.050	-0.7766 0.4795E-02	-0.5084 0.4718E-02	-0.4225 0.4350E-02	-0.3555 0.3751E-02	-0.3112 0.3817E-02	-0.2609 0.4041E-02	-0.2129 0.3777E-02	-0.1591 0.5198E-02
0.100	-0.5624 0.2447E-02	-0.3827 0.3033E-02	-0.3219 0.3544E-02	-0.2666 0.2039E-02	-0.2349 0.4126E-02	-0.2009 0.1916E-02	-0.1643 0.3657E-02	-0.1193 0.4478E-02
0.250	-0.2673 0.1570E-02	-0.1865 0.3736E-02	-0.1534 0.2744E-02	-0.1326 0.3863E-02	-0.1214 0.3145E-02	-0.9865E-01 0.2747E-02	-0.8363E-01 0.2649E-02	-0.5756E-01 0.3635E-02
0.500	0.4487E-02 0.2059E-02	-0.2148E-02 0.1909E-02	0.8903E-03 0.2165E-02	-0.1526E-02 0.1833E-02	0.4897E-03 0.3865E-02	0.4867E-02 0.2331E-02	0.4764E-03 0.2672E-02	0.2130E-03 0.2093E-02
0.750	0.2737 0.2479E-02	0.1866 0.3250E-02	0.1560 0.4664E-02	0.1320 0.2572E-02	0.1224 0.3128E-02	0.1027 0.2923E-02	0.8624E-01 0.3425E-02	0.5942E-01 0.1449E-02
0.900	0.5673 0.3838E-02	0.3858 0.3668E-02	0.3113 0.5464E-02	0.2677 0.3341E-02	0.2335 0.3201E-02	0.1960 0.3145E-02	0.1636 0.2939E-02	0.1176 0.3052E-02
0.950	0.7743 0.5899E-02	0.5140 0.5460E-02	0.4246 0.8173E-02	0.3569 0.6727E-02	0.3123 0.4748E-02	0.2564 0.4627E-02	0.2194 0.3715E-02	0.1526 0.2956E-02
0.975	0.9796 0.7414E-02	0.6392 0.1291E-01	0.5127 0.1213E-01	0.4388 0.9658E-02	0.3796 0.7875E-02	0.3159 0.4543E-02	0.2745 0.5070E-02	0.1866 0.5383E-02
0.990	1.245 0.1581E-01	0.7861 0.1199E-01	0.6419 0.1123E-01	0.5476 0.1170E-01	0.4653 0.7852E-02	0.3969 0.7801E-02	0.3247 0.3861E-02	0.2283 0.3404E-02

MEAN OF REGRESSION ON AVERAGES - COEFFICIENTS:	-0.546793E-03 0.360565E-02	0.345632 0.965110	-26.8065 61.1860	493.163 985.488
STD DEV OF REGRESSION - COEFFICIENTS:	0.821726E-03 0.364512E-03	2.33986 0.114357	160.953 8.46102	2717.16 144.377
REGRESSION ON VARIANCE - COEFFICIENTS:	3.03867 0.738746	31.1905 20.4323	-216.931 178.371	633.562 478.630

ESTIMATOR: SAMPLE MEDIAN; MU=0.0

*** WIDEST Y VALUES FOUND: YMIN=-3.878 YMAX= 2.972

TABLE III.E.2.6

SIMTBED Summary Statistics for Estimating μ by m in the BELAR(1) Process with α=.844 and γ=.89986

SUMMARY STATISTICS (MEAN/STD)   10 REPETITIONS

SUBSAMPLE SIZE	25	50	75	100	125	175	250	500
MEAN	0.1824E-02 / 0.3199E-02	-0.4160E-03 / 0.3438E-02	-0.3339E-02 / 0.4272E-02	0.3415E-02 / 0.7278E-02	-0.1755E-02 / 0.3971E-02	-0.3322E-02 / 0.4114E-02	-0.2433E-04 / 0.6343E-02	-0.1790E-02 / 0.5686E-02
STD	0.9836 / 0.5401E-02	0.7178 / 0.5020E-02	0.5684 / 0.4975E-02	0.4920 / 0.5128E-02	0.4202 / 0.4547E-02	0.3430 / 0.3500E-02	0.2823 / 0.3799E-02	0.1960 / 0.3789E-02
SKEWNESS	-0.6092E-01 / 0.6430E-01	0.3249E-01 / 0.5259E-01	-0.1587E-01 / 0.9316E-01	0.1209 / 0.7976E-01	0.3092E-02 / 0.4411E-01	-0.4456E-01 / 0.5816E-01	-0.7241E-01 / 0.6289E-01	-0.3148E-01 / 0.8714E-01
KURTOSIS	3.755 / 0.3174	3.094 / 0.2843	2.863 / 0.3610	1.785 / 0.1620	1.242 / 0.1040	1.320 / 0.2276	0.7943 / 0.1908	0.6159 / 0.7659E-01
SER.COR.	0.5547E-02 / 0.4386E-02	0.8988E-02 / 0.6316E-02	-0.7671E-02 / 0.9166E-02	-0.1157E-01 / 0.1024E-01	0.7063E-02 / 0.1224E-01	0.6001E-02 / 0.1070E-01	-0.1857E-01 / 0.1583E-01	0.6350E-03 / 0.2753E-01
QUANTILES								
0.010	-2.762 / 0.2397E-01	-1.969 / 0.3309E-01	-1.562 / 0.2947E-01	-1.262 / 0.4267E-01	-1.085 / 0.1884E-01	-0.9629 / 0.3439E-01	-0.7376 / 0.3408E-01	-0.5009 / 0.2583E-01
0.025	-2.088 / 0.2445E-01	-1.486 / 0.2398E-01	-1.175 / 0.1422E-01	-0.9509 / 0.2164E-01	-0.8714 / 0.1105E-01	-0.7271 / 0.2201E-01	-0.5857 / 0.1427E-01	-0.3989 / 0.1745E-01
0.050	-1.586 / 0.1507E-01	-1.144 / 0.9586E-02	-0.9137 / 0.1308E-01	-0.7759 / 0.1926E-01	-0.6951 / 0.1418E-01	-0.5652 / 0.9871E-02	-0.4761 / 0.1071E-01	-0.3123 / 0.1127E-01
0.100	-1.096 / 0.4830E-02	-0.8270 / 0.4083E-02	-0.6635 / 0.9367E-02	-0.5663 / 0.1266E-01	-0.5081 / 0.8365E-02	-0.4223 / 0.1030E-01	-0.3549 / 0.5633E-02	-0.2263 / 0.8793E-02
0.250	-0.4777 / 0.4984E-02	-0.3712 / 0.4212E-02	-0.3064 / 0.3472E-02	-0.2738 / 0.6955E-02	-0.2432 / 0.2609E-02	-0.1990 / 0.5540E-02	-0.1698 / 0.6234E-02	-0.1199 / 0.6699E-02
0.500	0.5166E-02 / 0.2544E-02	0.1421E-02 / 0.3687E-02	-0.3266E-02 / 0.3458E-02	-0.5486E-02 / 0.4454E-02	-0.3635E-03 / 0.4656E-02	-0.1122E-02 / 0.3143E-02	0.3458E-02 / 0.5228E-02	-0.1531E-02 / 0.6533E-02
0.750	0.4842 / 0.5230E-02	0.3768 / 0.5636E-02	0.3049 / 0.4400E-02	0.2736 / 0.6637E-02	0.2444 / 0.6249E-02	0.1976 / 0.4737E-02	0.1772 / 0.7982E-02	0.1135 / 0.5652E-02
0.900	1.095 / 0.9392E-02	0.8135 / 0.1063E-01	0.6585 / 0.1110E-01	0.5882 / 0.1357E-01	0.5019 / 0.1287E-01	0.4080 / 0.6081E-02	0.3498 / 0.9896E-02	0.2351 / 0.7893E-02
0.950	1.567 / 0.1650E-01	1.152 / 0.1478E-01	0.9138 / 0.1658E-01	0.7994 / 0.1794E-01	0.6904 / 0.1212E-01	0.5646 / 0.1206E-01	0.4530 / 0.1287E-01	0.3052 / 0.1391E-01
0.975	2.046 / 0.1600E-01	1.487 / 0.2031E-01	1.169 / 0.2637E-01	1.003 / 0.2115E-01	0.8699 / 0.1724E-01	0.7017 / 0.1666E-01	0.5633 / 0.2087E-01	0.3979 / 0.1728E-01
0.990	2.750 / 0.4207E-01	1.969 / 0.2562E-01	1.515 / 0.3523E-01	1.312 / 0.3706E-01	1.125 / 0.2240E-01	0.8783 / 0.2164E-01	0.7016 / 0.2832E-01	0.4807 / 0.1698E-01

MEAN OF REGRESSION ON AVERAGES - COEFFICIENTS:	-0.219128E-02 / 0.777981E-02	0.285343 / 1.71833	-17.2998 / 104.215	317.068 / 1645.87
STD DEV OF REGRESSION - COEFFICIENTS:	0.200120E-01 / 0.155958E-02	5.67426 / 0.419389	377.369 / 25.9857	6252.41 / 417.202
REGRESSION ON VARIANCE - COEFFICIENTS:	13.6139 / 3.08586	70.1572 / 69.5686	584.341 / 779.406	-3352.77 / 2046.38

ESTIMATOR: SAMPLE MEDIAN; MU=0.0
*** WIDEST Y VALUES FOUND: YMIN=-9.919   YMAX= 8.433

TABLE III.E.2.7

Efficiency of $\overline{X}$ Relative to m in BELAR(1) for $\gamma > 0$

N	$\gamma = +.1766$[1]	$\gamma = +.63662$	$\gamma = +.9$
25	.64	.69	.98
50	.58	.58	.81
75	.55	.55	.73
100	.54	.52	.67
125	.52	.50	.62
175	.53	.49	.57
250	.51	.47	.53
500	.50	.47	.48

1. For $\gamma = +.1766$ the results are based on a Type III experiment. For the other two cases, the results are based on Type II experiments.

We also simulated $\overline{X}$ and m for negatively correlated data from the BELAR(1) process. Type III simulations were used for $\overline{X}$ and m at $\gamma = -.63662$ and a Type II simulation for $\overline{X}$ at $\gamma = -.9$. From the Summary Statistics for $\overline{X}$ in Tables III.E.2.8 and III.E.2.9, we see $\overline{X}$ is unbiased and approximately Normal for sample sizes greater than 125. Estimates for the coefficients for the asymptotic variance are not significantly different from the theoretical values of .4441 and .1053.

From Table III.E.2.10, the most obvious point to be made is that even for moderately negatively correlated data, m is not Normally distributed even for subsamples of size 500. The sample median is unbiased, but the kurtosis is not decreasing fast enough. The variance of the sample median even at N = 500 is almost certainly not $(1/N)(1+\gamma/1-\gamma)$. However, the leading coefficient in the expansion for

the asymptotic variance is within a standard deviation of the hypothetical values $(1/N)(1+\gamma/1-\gamma)$. This would indicate, for the case of negative correlation, a much slower convergence of the sample median to Normality than for positively correlated data.

For negatively correlated data from the BELAR(1) process, we have observed that $\bar{X}$ does not lose efficiency relative to m as fast as for non-negatively correlated data. In fact, from Tables III.E.2.8 and III.E.2.10, it is clear that the variance of $\bar{X}$ is smaller than m for subsample size $N \leq 100$.

3. Estimators of Scale

a. Introduction

In the case of estimating the scale parameter, λ, we considered three estimators. Since $\mathrm{Var}(X_n) = 2\lambda^2$, we considered $\hat{\lambda}_1 = S/\sqrt{2}$ where

$$S^2 = \frac{1}{N} \sum_{i=1}^{N} (X_i - \bar{X})^2. \qquad \text{(III.E.3.1)}$$

Since the maximum likelihood estimator of λ for an i.i.d. sample with marginal Laplace distribution is the sample mean absolute deviation about the median, we set

$$\hat{\lambda}_2 = \frac{1}{N} \sum_{i=1}^{N} |X_i - m|. \qquad \text{(III.E.3.2)}$$

TABLE III.E.2.8

SIMTBED Summary Statistics for Estimating μ by $\overline{X}$ in the BELAR(1) Process with α=.5 and γ=-.63662

SUMMARY STATISTICS (MEAN/STD)   10 REPETITIONS

SUBSAMPLE SIZE	25	50	75	100	125	175	250	500
MEAN	-0.3649E-04 / 0.5454E-03	0.4706E-03 / 0.3687E-03	-0.4229E-03 / 0.4839E-03	-0.3605E-05 / 0.4178E-03	0.6060E-04 / 0.3146E-03	-0.6646E-03 / 0.4103E-03	0.3350E-03 / 0.5179E-03	0.3682E-04 / 0.3078E-03
STD	0.1386 / 0.3791E-03	0.9566E-01 / 0.3491E-03	0.7761E-01 / 0.2634E-03	0.6816E-01 / 0.2793E-03	0.5994E-01 / 0.2716E-03	0.5049E-01 / 0.3267E-03	0.4220E-01 / 0.3152E-03	0.3003E-01 / 0.3737E-03
SKEWNESS	0.1569E-01 / 0.6228E-02	-0.1531E-01 / 0.1049E-01	0.6558E-02 / 0.1394E-01	0.9900E-02 / 0.1219E-01	0.1277E-01 / 0.1565E-01	-0.3892E-01 / 0.3045E-01	-0.7610E-02 / 0.1653E-01	-0.8085E-01 / 0.2469E-01
KURTOSIS	0.3490 / 0.2602E-01	0.1997 / 0.2131E-01	0.1494 / 0.3972E-01	0.1166 / 0.2858E-01	0.1488E-01 / 0.3920E-01	0.1211 / 0.8270E-01	0.2763E-02 / 0.4908E-01	-0.8308E-02 / 0.5031E-01
SER.COR.	-0.3604E-02 / 0.4562E-02	0.2474E-02 / 0.4776E-02	-0.2129E-02 / 0.5660E-02	-0.1013E-01 / 0.8575E-02	0.5061E-02 / 0.4273E-02	-0.4530E-02 / 0.1269E-01	-0.1230E-01 / 0.1103E-01	-0.4168E-02 / 0.1687E-01
QUANTILES								
0.010	-0.3329 / 0.1499E-02	-0.2300 / 0.1138E-02	-0.1832 / 0.1975E-02	-0.1604 / 0.2217E-02	-0.1400 / 0.1263E-02	-0.1226 / 0.2455E-02	-0.9860E-01 / 0.1698E-02	-0.7315E-01 / 0.1953E-02
0.025	-0.2746 / 0.1366E-02	-0.1890 / 0.9175E-03	-0.1543 / 0.1185E-02	-0.1345 / 0.1225E-02	-0.1179 / 0.9601E-03	-0.1011 / 0.1124E-02	-0.8238E-01 / 0.1185E-02	-0.5935E-01 / 0.7617E-03
0.050	-0.2269 / 0.1516E-02	-0.1573 / 0.5621E-03	-0.1281 / 0.7780E-03	-0.1126 / 0.7481E-03	-0.9959E-01 / 0.8235E-03	-0.8396E-01 / 0.7673E-03	-0.6937E-01 / 0.1168E-02	-0.4970E-01 / 0.7705E-03
0.100	-0.1744 / 0.1159E-02	-0.1210 / 0.6068E-03	-0.9946E-01 / 0.5743E-03	-0.8682E-01 / 0.7445E-03	-0.7717E-01 / 0.6049E-03	-0.6512E-01 / 0.9794E-03	-0.5370E-01 / 0.9651E-03	-0.3907E-01 / 0.8283E-03
0.250	-0.8998E-01 / 0.5760E-03	-0.6223E-01 / 0.4510E-03	-0.5215E-01 / 0.7097E-03	-0.4546E-01 / 0.5620E-03	-0.4024E-01 / 0.7492E-03	-0.3390E-01 / 0.7438E-03	-0.2824E-01 / 0.6127E-03	-0.2031E-01 / 0.5989E-03
0.500	0.1085E-04 / 0.6387E-03	0.2883E-03 / 0.4715E-03	-0.1001E-02 / 0.6735E-03	-0.3133E-03 / 0.5596E-03	-0.2074E-03 / 0.4404E-03	-0.2971E-03 / 0.3634E-03	0.1668E-03 / 0.4821E-03	0.6234E-03 / 0.2948E-03
0.750	0.8962E-01 / 0.6382E-03	0.6326E-01 / 0.6082E-03	0.5155E-01 / 0.4084E-03	0.4534E-01 / 0.6948E-03	0.4018E-01 / 0.4818E-03	0.3213E-01 / 0.5663E-03	0.2905E-01 / 0.6375E-03	0.2054E-01 / 0.3659E-03
0.900	0.1742 / 0.7265E-03	0.1225 / 0.7844E-03	0.9777E-01 / 0.6867E-03	0.8663E-01 / 0.5293E-03	0.7691E-01 / 0.6851E-03	0.6395E-01 / 0.7632E-03	0.5368E-01 / 0.6418E-03	0.3869E-01 / 0.7445E-03
0.950	0.2268 / 0.1257E-02	0.1580 / 0.1081E-02	0.1272 / 0.8516E-03	0.1131 / 0.7150E-03	0.9900E-01 / 0.5621E-03	0.8222E-01 / 0.7288E-03	0.6974E-01 / 0.8863E-03	0.4816E-01 / 0.9911E-03
0.975	0.2768 / 0.1378E-02	0.1887 / 0.1606E-02	0.1528 / 0.1169E-02	0.1353 / 0.1233E-02	0.1181 / 0.1247E-02	0.9803E-01 / 0.1515E-02	0.8273E-01 / 0.1098E-02	0.5853E-01 / 0.1881E-02
0.990	0.3357 / 0.1701E-02	0.2263 / 0.2032E-02	0.1842 / 0.1512E-02	0.1601 / 0.1084E-02	0.1409 / 0.2217E-02	0.1174 / 0.2316E-02	0.1012 / 0.1399E-02	0.6743E-01 / 0.1636E-02

MEAN OF REGRESSION ON AVERAGES - COEFFICIENTS:	0.459609E-03 / 0.421741E-03	-0.117524 / 0.113401	13.0443 / 7.84241	-224.269 / 134.738
STD DEV OF REGRESSION - COEFFICIENTS:	0.217856E-03 / 0.112211E-03	0.584499 / 0.393611E-01	38.9123 / 2.71227	650.576 / 46.3178
REGRESSION ON VARIANCE - COEFFICIENTS:	0.418171 / 0.418209E-01	0.847251 / 1.11487	-7.18129 / 9.34865	22.4744 / 24.2949

ESTIMATOR: SAMPLE MEAN; MU=0.0

*** WIDEST Y VALUES FOUND: YMIN=-.7151 YMAX=0.7047

TABLE III.E.2.9

SIMTBED Summary Statistics for Estimating μ by $\overline{X}$ in the BELAR(1) Process with α=.844 and γ=-.89986

SUMMARY STATISTICS (MEAN/STD) 10 REPETITIONS

SUBSAMPLE SIZE	25	50	75	100	125	175	250	500
MEAN	0.3108E-03 0.2748E-03	-0.2264E-03 0.1899E-03	-0.1530E-03 0.2848E-03	0.6019E-04 0.5204E-03	-0.1936E-03 0.3685E-03	-0.1092E-03 0.3552E-03	-0.1315E-03 0.4816E-03	0.1675E-03 0.3641E-03
STD	0.7716E-01 0.3755E-03	0.5006E-01 0.2061E-03	0.3973E-01 0.3102E-03	0.3419E-01 0.2885E-03	0.3018E-01 0.3228E-03	0.2498E-01 0.1799E-03	0.2102E-01 0.2700E-03	0.1482E-01 0.2274E-03
SKEWNESS	-0.1161E-01 0.3089E-01	-0.4935E-01 0.2174E-01	0.8011E-02 0.2414E-01	0.6199E-02 0.3300E-01	-0.1386E-02 0.3091E-01	-0.3600E-01 0.2631E-01	-0.2336E-01 0.3953E-01	0.6347E-01 0.5314E-01
KURTOSIS	1.257 0.8913E-01	0.7315 0.6563E-01	0.4066 0.8248E-01	0.2986 0.4599E-01	0.3417 0.5215E-01	0.1522 0.5532E-01	-0.3240E-01 0.7474E-01	-0.1436E-01 0.8464E-01
SER.COR.	0.1654E-03 0.5544E-02	0.7664E-02 0.5429E-02	-0.3939E-02 0.5164E-02	-0.4805E-02 0.7955E-02	0.7600E-02 0.8020E-02	-0.5481E-02 0.1134E-01	-0.1450E-01 0.1772E-01	-0.2447E-01 0.2907E-01
QUANTILES								
0.010	-0.1979 0.3037E-02	-0.1255 0.1399E-02	-0.9881E-01 0.1816E-02	-0.9387E-01 0.1760E-02	-0.7286E-01 0.1331E-02	-0.6105E-01 0.1086E-02	-0.5017E-01 0.1619E-02	-0.3437E-01 0.8037E-03
0.025	-0.1565 0.6130E-03	-0.1018 0.9853E-03	-0.7952E-01 0.1420E-02	-0.6846E-01 0.1139E-02	-0.6042E-01 0.8393E-03	-0.5089E-01 0.7422E-03	-0.4206E-01 0.1093E-02	-0.2870E-01 0.8412E-03
0.050	-0.1248 0.8323E-03	-0.8182E-01 0.8024E-03	-0.6516E-01 0.1108E-02	-0.5615E-01 0.9935E-03	-0.4943E-01 0.8700E-03	-0.4185E-01 0.6087E-03	-0.3627E-01 0.7372E-03	-0.2445E-01 0.6691E-03
0.100	-0.9206E-01 0.4205E-03	-0.6227E-01 0.8090E-03	-0.5053E-01 0.6299E-03	-0.4323E-01 0.8762E-03	-0.3829E-01 0.7313E-03	-0.3155E-01 0.3519E-03	-0.2732E-01 0.5934E-03	-0.1863E-01 0.6385E-03
0.250	-0.4579E-01 0.3998E-03	-0.3158E-01 0.3264E-03	-0.2575E-01 0.3306E-03	-0.2191E-01 0.5587E-03	-0.2004E-01 0.4737E-03	-0.1646E-01 0.3753E-03	-0.1410E-01 0.5746E-03	-0.1001E-01 0.5991E-03
0.500	0.3652E-03 0.3575E-03	-0.4574E-03 0.2168E-03	-0.2735E-04 0.2063E-03	-0.1215E-03 0.6108E-03	-0.2266E-03 0.3463E-03	0.2259E-05 0.5214E-03	0.6179E-04 0.4194E-03	-0.8280E-04 0.4632E-03
0.750	0.4616E-01 0.5552E-03	0.3178E-01 0.3010E-03	0.2587E-01 0.3600E-03	0.2227E-01 0.6092E-03	0.1969E-01 0.4893E-03	0.1649E-01 0.4600E-03	0.1401E-01 0.5481E-03	0.1005E-01 0.4324E-03
0.900	0.9366E-01 0.6910E-03	0.6205E-01 0.3087E-03	0.4952E-01 0.6459E-03	0.4292E-01 0.7479E-03	0.3767E-01 0.6082E-03	0.3209E-01 0.4327E-03	0.2670E-01 0.9578E-03	0.1883E-01 0.6453E-03
0.950	0.1266 0.1437E-02	0.8162E-01 0.6596E-03	0.6531E-01 0.9030E-03	0.5706E-01 0.6790E-03	0.4854E-01 0.1098E-02	0.4094E-01 0.6842E-03	0.3500E-01 0.1144E-02	0.2542E-01 0.7898E-03
0.975	0.1570 0.1378E-02	0.9870E-01 0.1083E-02	0.7818E-01 0.1258E-02	0.6901E-01 0.7734E-03	0.5998E-01 0.1095E-02	0.4902E-01 0.1065E-02	0.4133E-01 0.9002E-03	0.3203E-01 0.9031E-03
0.990	0.1980 0.1802E-02	0.1219 0.9874E-03	0.9609E-01 0.1307E-02	0.8197E-01 0.1431E-02	0.7414E-01 0.1606E-02	0.5839E-01 0.1494E-02	0.4952E-01 0.1321E-02	0.3655E-01 0.1452E-02

MEAN OF REGRESSION ON AVERAGES - COEFFICIENTS:	0.109168E-03 0.431953E-03	-0.321683E-01 0.106441	0.808037 6.97772	1.01668 115.848
STD DEV OF REGRESSION - COEFFICIENTS:	0.158562E-02 0.151984E-03	0.447247 0.401888E-01	29.5613 2.54413	490.719 41.3060
REGRESSION ON VARIANCE - COEFFICIENTS:	0.107444 0.205764E-01	-0.912619E-03 0.551045	0.590624 4.52098	2.24537 11.5775

ESTIMATOR: SAMPLE MEAN; MU=0.0

*** WIDEST Y VALUES FOUND: YMIN=-.4513 YMAX=0.4718

TABLE III.E.2.10

SIMTBED Summary Statistics for Estimating μ by m in the BELAR(1) Process with α=.5 and γ=-.63662

SUMMARY STATISTICS (MEAN/STD)   10 REPETITIONS

SUBSAMPLE SIZE	25	50	75	100	125	175	250	500
MEAN	-0.1126E-03 0.5892E-03	0.3257E-03 0.3433E-03	-0.1253E-03 0.4073E-03	0.2723E-03 0.4672E-03	0.1518E-03 0.4888E-03	-0.3766E-03 0.4609E-03	0.3109E-03 0.3999E-03	0.2271E-04 0.4375E-03
STD	0.1529 0.6224E-03	0.9930E-01 0.4012E-03	0.7969E-01 0.2521E-03	0.6714E-01 0.3630E-03	0.5927E-01 0.2813E-03	0.4892E-01 0.1362E-03	0.4079E-01 0.3269E-03	0.2785E-01 0.3288E-03
SKEWNESS	-0.3567E-01 0.1583E-01	-0.7905E-02 0.1176E-01	-0.2203E-01 0.2748E-01	0.3989E-02 0.1550E-01	0.1947E-01 0.1315E-01	-0.5853E-01 0.2869E-01	-0.1175E-01 0.4838E-01	0.4283E-01 0.3318E-01
KURTOSIS	1.827 0.6533E-01	0.9781 0.7180E-01	0.8004 0.4909E-01	0.5661 0.4429E-01	0.4535 0.4716E-01	0.3246 0.4478E-01	0.4211 0.8480E-01	0.2186 0.5933E-01
SER.COR.	-0.3788E-05 0.2746E-02	0.7386E-02 0.3700E-02	-0.2001E-02 0.5540E-02	0.5934E-02 0.8320E-02	0.4612E-02 0.8254E-02	-0.9020E-03 0.1076E-01	0.7516E-02 0.1073E-01	-0.1740E-01 0.2204E-01
QUANTILES								
0.010	-0.4207 0.3183E-02	-0.2497 0.2206E-02	-0.1979 0.2250E-02	-0.1649 0.2449E-02	-0.1452 0.1697E-02	-0.1211 0.9701E-03	-0.9787E-01 0.2562E-02	-0.6651E-01 0.1771E-02
0.025	-0.3277 0.2295E-02	-0.2029 0.1199E-02	-0.1615 0.1063E-02	-0.1354 0.1524E-02	-0.1187 0.1319E-02	-0.9845E-01 0.1135E-02	-0.8197E-01 0.1656E-02	-0.5705E-01 0.1391E-02
0.050	-0.2587 0.1639E-02	-0.1633 0.5834E-03	-0.1312 0.9124E-03	-0.1104 0.1017E-02	-0.9917E-01 0.9721E-03	-0.8190E-01 0.9570E-03	-0.6806E-01 0.1080E-02	-0.4649E-01 0.1016E-02
0.100	-0.1894 0.1059E-02	-0.1210 0.9843E-03	-0.9867E-01 0.5579E-03	-0.8329E-01 0.8449E-03	-0.7433E-01 0.6785E-03	-0.6314E-01 0.7424E-03	-0.5153E-01 0.7830E-03	-0.3479E-01 0.8256E-03
0.250	-0.9037E-01 0.5789E-03	-0.5967E-01 0.6801E-03	-0.4972E-01 0.4950E-03	-0.4225E-01 0.4542E-03	-0.3696E-01 0.5147E-03	-0.3104E-01 0.6065E-03	-0.2537E-01 0.4353E-03	-0.1795E-01 0.4750E-03
0.500	-0.4159E-03 0.5182E-03	0.5310E-03 0.3965E-03	-0.1725E-03 0.5441E-03	0.4302E-03 0.6609E-03	-0.2512E-03 0.5233E-03	-0.8457E-04 0.6572E-03	0.9294E-03 0.3471E-03	-0.2354E-03 0.3330E-03
0.750	0.9048E-01 0.7797E-03	0.6045E-01 0.6429E-03	0.4890E-01 0.5477E-03	0.4215E-01 0.4856E-03	0.3791E-01 0.5771E-03	0.3089E-01 0.3609E-03	0.2682E-01 0.4083E-03	0.1845E-01 0.5420E-03
0.900	0.1892 0.1346E-02	0.1227 0.5991E-03	0.9766E-01 0.5615E-03	0.8355E-01 0.7445E-03	0.7458E-01 0.8901E-03	0.6135E-01 0.6171E-03	0.5082E-01 0.9694E-03	0.3506E-01 0.1096E-02
0.950	0.2612 0.1963E-02	0.1636 0.9261E-03	0.1311 0.8512E-03	0.1116 0.8944E-03	0.9821E-01 0.1168E-02	0.8051E-01 0.1196E-02	0.6693E-01 0.1163E-02	0.4704E-01 0.1164E-02
0.975	0.3306 0.1931E-02	0.2023 0.1460E-02	0.1625 0.1615E-02	0.1375 0.9410E-03	0.1193 0.9515E-03	0.9744E-01 0.1272E-02	0.8183E-01 0.1524E-02	0.5885E-01 0.1222E-02
0.990	0.4185 0.3015E-02	0.2494 0.1733E-02	0.1977 0.2534E-02	0.1653 0.1895E-02	0.1442 0.1558E-02	0.1163 0.2228E-02	0.9906E-01 0.2531E-02	0.6875E-01 0.1754E-02

MEAN OF REGRESSION ON AVERAGES - COEFFICIENTS:	0.291504E-03 0.512992E-03	-0.822573E-01 0.152727	6.61271 10.8829	-120.191 186.871
STD DEV OF REGRESSION - COEFFICIENTS:	0.201131E-02 0.204604E-03	0.566700 0.631862E-01	38.4287 4.47829	648.563 75.3094
REGRESSION ON VARIANCE - COEFFICIENTS:	0.249473 0.270036E-01	4.14361 0.725145	-31.3516 6.38625	191.991 17.3973

ESTIMATOR: SAMPLE MEDIAN; MU=0.0

*** WIDEST Y VALUES FOUND: YMIN=-1.065 YMAX= 1.013

As a third alternative, we chose the scaled median absolute deviation about the median,

$$\hat{\lambda}_3 = \operatorname*{med}_i \left\{ \frac{|X_i - m|}{.69315} \right\}. \qquad \text{(III.E.3.3)}$$

The scaled median absolute deviation is a frequently used robust estimator of scale [Ref. 38]. In the simulations, we assumed that X_n are Laplace with median = mean = 0 for all n. Table III.E.3.1 contains a summary of the type simulation (as defined in Table III.E.1.1), the estimator($\hat{\lambda}_1, \hat{\lambda}_2, \hat{\lambda}_3$) and the values of α and γ that were used.

TABLE III.E.3.1

Summary of Simulation Schedule for Estimators of λ

γ	-.89986	.17664	.63662
α	.844	.1	.5
Estimator			
$\hat{\lambda}_1$	Type II	Type III	Type I
$\hat{\lambda}_2$	Type II	Type III	Type I
$\hat{\lambda}_3$	Type II	Type III	Type I

b. Simulation Results

In the Type III simulation (See Tables III.E.3.2 - III.E.3.4), using slightly correlated (γ = .17664) realizations of the BELAR(1) process, we found the best estimator of λ to be $\hat{\lambda}_2$, the sample mean absolute deviation. It appears to be unbiased for all subsample sizes. The skewness and kurtosis are decreasing with increased sample

TABLE III.E.3.2

SIMTBED Summary Statistics for Estimating λ by $\hat{\lambda}_1$ in the BELAR(1) Process with α=.1 and γ=.17664

SUMMARY STATISTICS (MEAN/STD) 10 REPETITIONS

SUBSAMPLE SIZE	25	50	75	100	125	175	250	500
MEAN	0.9720 0.6905E-03	0.9857 0.9033E-03	0.9893 0.1227E-02	0.9923 0.9711E-03	0.9937 0.1155E-02	0.9960 0.8674E-03	0.9960 0.8299E-03	0.9993 0.8551E-03
STD	0.2306 0.8126E-03	0.1678 0.6423E-03	0.1386 0.6246E-03	0.1212 0.6784E-03	0.1095 0.5398E-03	0.9252E-01 0.6459E-03	0.7791E-01 0.7681E-03	0.5629E-01 0.6052E-03
SKEWNESS	0.6843 0.1621E-01	0.5224 0.1551E-01	0.4353 0.1451E-01	0.4032 0.2548E-01	0.3578 0.2609E-01	0.3210 0.2302E-01	0.2665 0.2820E-01	0.2144 0.3495E-01
KURTOSIS	0.8585 0.5894E-01	0.6271 0.7811E-01	0.3867 0.3256E-01	0.3990 0.1124	0.3620 0.7669E-01	0.1666 0.6398E-01	0.8492E-01 0.8452E-01	0.1551 0.1180
SER.COR.	-0.1085E-02 0.4023E-02	-0.3902E-02 0.4137E-02	-0.7923E-03 0.6271E-02	-0.5706E-02 0.6948E-02	0.2392E-02 0.7901E-02	0.3349E-03 0.1175E-01	-0.9790E-02 0.9095E-02	0.2062E-01 0.1464E-01
QUANTILES								
0.010	0.5401 0.2265E-02	0.6519 0.2449E-02	0.7082 0.2191E-02	0.7427 0.2517E-02	0.7604 0.2754E-02	0.8004 0.3895E-02	0.8296 0.1759E-02	0.8742 0.3377E-02
0.025	0.5917 0.1934E-02	0.6956 0.2079E-02	0.7444 0.1261E-02	0.7760 0.1155E-02	0.7947 0.1985E-02	0.8299 0.1554E-02	0.8530 0.1423E-02	0.8933 0.1991E-02
0.050	0.6391 0.6828E-03	0.7350 0.1485E-02	0.7773 0.1492E-02	0.8062 0.1298E-02	0.8243 0.1598E-02	0.8517 0.1741E-02	0.8741 0.1509E-02	0.9106 0.1055E-02
0.100	0.7003 0.9766E-03	0.7832 0.1113E-02	0.8195 0.1252E-02	0.8436 0.1248E-02	0.8566 0.1350E-02	0.8810 0.1014E-02	0.8988 0.1445E-02	0.9292 0.1080E-02
0.250	0.8095 0.1029E-02	0.8680 0.8235E-03	0.8832 0.1094E-02	0.9083 0.8915E-03	0.9187 0.9149E-03	0.9317 0.7772E-03	0.9420 0.9033E-03	0.9597 0.1611E-02
0.500	0.9486 0.7206E-03	0.9728 0.8235E-03	0.9797 0.1527E-02	0.9852 0.1571E-02	0.9879 0.1294E-02	0.9913 0.9321E-03	0.9930 0.9874E-03	0.9981 0.1045E-02
0.750	1.111 0.1040E-02	1.089 0.1244E-02	1.075 0.1492E-02	1.068 0.1611E-02	1.063 0.1485E-02	1.054 0.1534E-02	1.046 0.8795E-03	1.035 0.1151E-02
0.900	1.276 0.1595E-02	1.204 0.1669E-02	1.171 0.2169E-02	1.150 0.2234E-02	1.136 0.1679E-02	1.117 0.1381E-02	1.100 0.2236E-02	1.073 0.1568E-02
0.950	1.389 0.1887E-02	1.280 0.2436E-02	1.233 0.2604E-02	1.203 0.2884E-02	1.189 0.2297E-02	1.158 0.2012E-02	1.130 0.3036E-02	1.094 0.2046E-02
0.975	1.492 0.3189E-02	1.354 0.2100E-02	1.290 0.3683E-02	1.252 0.3883E-02	1.225 0.3492E-02	1.193 0.3788E-02	1.157 0.3555E-02	1.118 0.3184E-02
0.990	1.624 0.5867E-02	1.444 0.3637E-02	1.356 0.3349E-02	1.317 0.4819E-02	1.285 0.5110E-02	1.230 0.4384E-02	1.193 0.4863E-02	1.141 0.4283E-02

MEAN OF REGRESSION ON AVERAGES - COEFFICIENTS:	1.00102 0.144481E-02	-1.06694 0.427177	28.6613 28.9693	-288.711 490.479
STD DEV OF REGRESSION - COEFFICIENTS:	0.430973E-02 0.353764E-03	1.06405 0.738681E-01	68.4717 5.10782	1128.61 86.0285
REGRESSION ON VARIANCE - COEFFICIENTS:	1.68838 0.128403	-3.94571 3.78840	10.4502 39.4282	-20.9397 88.9123

ESTIMATOR: SQRT(.5*SUM((X-HMU)**2)) LMDA=1.
*** WIDEST Y VALUES FOUND: YMIN=0.2146 YMAX= 3.072

TABLE III.E.3.3

SIMTBED Summary Statistics for Estimating λ by $\hat{\lambda}_2$ in the BELAR(1) Process with α=.1 and γ=.17664

SUMMARY STATISTICS (MEAN/STD) 10 REPETITIONS

SUBSAMPLE SIZE	25	50	75	100	125	175	250	500
MEAN	0.9992 / 0.7634E-03	1.000 / 0.7351E-03	0.9995 / 0.9601E-03	1.000 / 0.9434E-03	0.9993 / 0.9874E-03	1.000 / 0.7279E-03	0.9989 / 0.8235E-03	1.001 / 0.8795E-03
STD	0.2707 / 0.6106E-03	0.1560 / 0.6606E-03	0.1274 / 0.4930E-03	0.1109 / 0.7270E-03	0.9978E-01 / 0.5010E-03	0.8361E-01 / 0.5326E-03	0.7009E-01 / 0.7822E-03	0.5938E-01 / 0.5718E-03
SKEWNESS	0.5495 / 0.1263E-01	0.3697 / 0.1257E-01	0.2850 / 0.1481E-01	0.2909 / 0.2039E-01	0.2592 / 0.2247E-01	0.2267 / 0.1985E-01	0.1833 / 0.1453E-01	0.1365 / 0.3395E-01
KURTOSIS	0.4962 / 0.3806E-01	0.1906 / 0.3162E-01	0.1081 / 0.2354E-01	0.2167 / 0.6442E-01	0.2527 / 0.8317E-01	0.3982E-01 / 0.3624E-01	0.6539E-02 / 0.4835E-01	0.4374E-01 / 0.7762E-01
SER.COR.	0.3035E-03 / 0.2977E-02	-0.2042E-02 / 0.4380E-02	-0.1049E-02 / 0.6510E-02	-0.1370E-01 / 0.7374E-02	0.7937E-02 / 0.7179E-02	-0.6041E-02 / 0.1208E-01	-0.1265E-01 / 0.1014E-01	0.2067E-01 / 0.1394E-01
QUANTILES								
0.010	0.5711 / 0.2274E-02	0.6791 / 0.1396E-02	0.7306 / 0.2079E-02	0.7636 / 0.1350E-02	0.7801 / 0.2363E-02	0.8203 / 0.2737E-02	0.8435 / 0.2140E-02	0.8871 / 0.3132E-02
0.025	0.6257 / 0.1873E-02	0.7224 / 0.1551E-02	0.7664 / 0.1430E-02	0.7975 / 0.1467E-02	0.8139 / 0.2343E-02	0.8453 / 0.1109E-02	0.8687 / 0.1598E-02	0.9038 / 0.1558E-02
0.050	0.6741 / 0.9820E-03	0.7607 / 0.1426E-02	0.8003 / 0.1448E-02	0.8259 / 0.8915E-03	0.8412 / 0.1411E-02	0.8681 / 0.1155E-02	0.8889 / 0.1685E-02	0.9204 / 0.1318E-02
0.100	0.7333 / 0.8735E-03	0.8071 / 0.9766E-03	0.8401 / 0.1366E-02	0.8611 / 0.1235E-02	0.8748 / 0.6429E-03	0.8961 / 0.1231E-02	0.9113 / 0.1330E-02	0.9375 / 0.1415E-02
0.250	0.8430 / 0.8551E-03	0.8905 / 0.7974E-03	0.9112 / 0.8974E-03	0.9239 / 0.9033E-03	0.9313 / 0.8040E-03	0.9429 / 0.8855E-03	0.9500 / 0.1014E-02	0.9657 / 0.1123E-02
0.500	0.9812 / 0.1003E-02	0.9911 / 0.7351E-03	0.9938 / 0.8105E-03	0.9951 / 0.1322E-02	0.9950 / 0.1113E-02	0.9967 / 0.8551E-03	0.9972 / 0.7494E-03	1.000 / 0.8795E-03
0.750	1.137 / 0.1290E-02	1.100 / 0.1123E-02	1.082 / 0.1277E-02	1.072 / 0.1265E-02	1.064 / 0.1346E-02	1.054 / 0.8235E-03	1.045 / 0.1248E-02	1.034 / 0.1847E-02
0.900	1.290 / 0.1698E-02	1.205 / 0.2239E-02	1.166 / 0.2306E-02	1.143 / 0.2100E-02	1.127 / 0.1611E-02	1.109 / 0.9321E-03	1.091 / 0.1338E-02	1.067 / 0.1155E-02
0.950	1.392 / 0.1747E-02	1.272 / 0.2329E-02	1.219 / 0.2836E-02	1.189 / 0.2100E-02	1.168 / 0.2288E-02	1.146 / 0.1688E-02	1.120 / 0.2858E-02	1.085 / 0.1075E-02
0.975	1.485 / 0.3542E-02	1.336 / 0.2604E-02	1.267 / 0.3163E-02	1.234 / 0.2557E-02	1.206 / 0.2243E-02	1.175 / 0.2174E-02	1.143 / 0.2950E-02	1.100 / 0.2334E-02
0.990	1.599 / 0.5093E-02	1.408 / 0.3971E-02	1.320 / 0.3706E-02	1.289 / 0.4585E-02	1.254 / 0.3963E-02	1.210 / 0.3576E-02	1.172 / 0.3469E-02	1.125 / 0.3316E-02

MEAN OF REGRESSION ON AVERAGES - COEFFICIENTS:	1.00067 / 0.129392E-02	-0.210906 / 0.385509	13.3010 / 25.9777	-209.509 / 437.424
STD DEV OF REGRESSION - COEFFICIENTS:	0.380346E-02 / 0.323496E-03	0.931218 / 0.740383E-01	59.8151 / 4.48457	988.645 / 69.0615
REGRESSION ON VARIANCE - COEFFICIENTS:	1.29063 / 0.143586	-1.91917 / 3.91770	4.22873 / 32.8819	-4.84233 / 84.7695

ESTIMATOR: SAMPLE MEAN ABS DEV; LMDA=1.0

*** WIDEST Y VALUES FOUND: YMIN=0.3192 YMAX= 2.381

TABLE III.E.3.4

SIMTBED Summary Statistics for Estimating λ by $\hat{\lambda}_3$ in the BELAR(1) Process with α=.1 and γ=.17664

SUMMARY STATISTICS (MEAN/STD) 10 REPETITIONS

SUBSAMPLE SIZE	25	50	75	100	125	175	250	500
MEAN	1.034 / 0.1477E-02	1.017 / 0.1231E-02	1.012 / 0.9490E-03	1.008 / 0.1055E-02	1.006 / 0.1075E-02	1.006 / 0.5823E-03	1.002 / 0.1085E-02	1.002 / 0.1370E-02
STD	0.3096 / 0.3175E-03	0.2151 / 0.7873E-03	0.1771 / 0.8052E-03	0.1521 / 0.8863E-03	0.1362 / 0.1113E-02	0.1151 / 0.8810E-03	0.9702E-01 / 0.1205E-02	0.6773E-01 / 0.8323E-03
SKEWNESS	0.6870 / 0.1153E-01	0.4864 / 0.1726E-01	0.3803 / 0.2948E-01	0.3461 / 0.2072E-01	0.2840 / 0.1583E-01	0.2861 / 0.2454E-01	0.2395 / 0.3477E-01	0.1300 / 0.4584E-01
KURTOSIS	0.7309 / 0.3713E-01	0.3442 / 0.5238E-01	0.2495 / 0.5565E-01	0.2286 / 0.6838E-01	0.1120 / 0.5323E-01	0.1118 / 0.5435E-01	0.1265 / 0.1029	0.8686E-01 / 0.1020
SER.COR.	-0.1376E-02 / 0.4104E-02	0.3735E-02 / 0.5758E-02	0.2187E-03 / 0.4280E-02	-0.1155E-01 / 0.9401E-02	0.9438E-02 / 0.9115E-02	-0.7615E-03 / 0.8388E-02	-0.8541E-03 / 0.8649E-02	0.8775E-03 / 0.1388E-01
QUANTILES								
0.010	0.4641 / 0.1688E-02	0.5915 / 0.1967E-02	0.6518 / 0.3226E-02	0.6949 / 0.2488E-02	0.7155 / 0.2536E-02	0.7636 / 0.3575E-02	0.7912 / 0.2926E-02	0.8504 / 0.4423E-02
0.025	0.5301 / 0.1530E-02	0.6446 / 0.1850E-02	0.6955 / 0.2007E-02	0.7347 / 0.1200E-02	0.7554 / 0.2290E-02	0.7972 / 0.3312E-02	0.8223 / 0.2286E-02	0.8693 / 0.3541E-02
0.050	0.5924 / 0.1744E-02	0.6958 / 0.1474E-02	0.7410 / 0.2374E-02	0.7760 / 0.1273E-02	0.7923 / 0.2215E-02	0.8292 / 0.1679E-02	0.8496 / 0.2590E-02	0.8923 / 0.2193E-02
0.100	0.6685 / 0.1370E-02	0.7557 / 0.1499E-02	0.7942 / 0.1762E-02	0.8212 / 0.1205E-02	0.8361 / 0.1744E-02	0.8604 / 0.1509E-02	0.8808 / 0.2302E-02	0.9150 / 0.2370E-02
0.250	0.8124 / 0.1298E-02	0.8631 / 0.1009E-02	0.8866 / 0.1029E-02	0.9000 / 0.1214E-02	0.9109 / 0.1366E-02	0.9252 / 0.1269E-02	0.9355 / 0.1118E-02	0.9557 / 0.1959E-02
0.500	0.9929 / 0.1541E-02	0.9995 / 0.1019E-02	1.002 / 0.1506E-02	1.000 / 0.1235E-02	0.9988 / 0.1070E-02	1.001 / 0.6346E-03	0.9983 / 0.1358E-02	1.000 / 0.1530E-02
0.750	1.219 / 0.2630E-02	1.151 / 0.1906E-02	1.127 / 0.1354E-02	1.106 / 0.1807E-02	1.094 / 0.1890E-02	1.081 / 0.1080E-02	1.065 / 0.1477E-02	1.047 / 0.1544E-02
0.900	1.445 / 0.2969E-02	1.303 / 0.2571E-02	1.242 / 0.2102E-02	1.206 / 0.2181E-02	1.183 / 0.2359E-02	1.155 / 0.1595E-02	1.129 / 0.2215E-02	1.089 / 0.1909E-02
0.950	1.598 / 0.2882E-02	1.399 / 0.3884E-02	1.318 / 0.3542E-02	1.271 / 0.2538E-02	1.241 / 0.3209E-02	1.204 / 0.2725E-02	1.166 / 0.3219E-02	1.118 / 0.2343E-02
0.975	1.737 / 0.4055E-02	1.487 / 0.3949E-02	1.395 / 0.5720E-02	1.330 / 0.4766E-02	1.289 / 0.3992E-02	1.247 / 0.4019E-02	1.204 / 0.4827E-02	1.140 / 0.3443E-02
0.990	1.910 / 0.5967E-02	1.600 / 0.4924E-02	1.476 / 0.7626E-02	1.401 / 0.4567E-02	1.356 / 0.5060E-02	1.300 / 0.4492E-02	1.253 / 0.7429E-02	1.168 / 0.4998E-02

	100	125	250	500
MEAN OF REGRESSION ON AVERAGES - COEFFICIENTS:	0.999350 / 0.161453E-02	0.979559 / 0.450446	-6.73084 / 29.8486	105.973 / 503.247
STD DEV OF REGRESSION - COEFFICIENTS:	0.510409E-02 / 0.258720E-03	1.32920 / 0.602317E-01	86.8768 / 4.40012	1438.11 / 78.6893
REGRESSION ON VARIANCE - COEFFICIENTS:	2.16808 / 0.330769	4.90821 / 0.88885	-46.8788 / 73.0212	140.153 / 184.292

ESTIMATOR: SAMPLE MEDIAN ABS DEV; LMDA=1.0

sizes. But even for N = 500, the skewness is still significantly different than 0. Using two-sided t-tests with 18 degrees of freedom for the equality of means of two Normal populations with unknown variances at the 90% confidence level, we reject each of the hypotheses independently that: (1) $Var(\hat{\lambda}_1) = Var(\hat{\lambda}_2)$ and (2) $Var(\hat{\lambda}_1) = Var(\hat{\lambda}_3)$. The mean relative asymptotic efficiency of $\hat{\lambda}_2$ and $\hat{\lambda}_3$ to $\hat{\lambda}_1$ are estimated from the regression on variance coefficients to be 76% for $\hat{\lambda}_1$ and 60% for $\hat{\lambda}_3$.

Both $\hat{\lambda}_1$ and $\hat{\lambda}_3$ appear from the simulation to be biased. From the second coefficient in the mean of regression on average in Table III.E.3.2, $\hat{\lambda}_1$ appears to have a negative bias of order(1/N). From Table III.E.3.4 it appears that $\hat{\lambda}_3$ has a positive bias of order(1/N). However, since the leading term in the expansion of the mean for both estimators is the true value of γ, it appears that both $\hat{\lambda}_1$ and $\hat{\lambda}_3$ are asymptotically unbiased.

When we considered moderately to highly correlated data (see Tables III.E.3.5 - III.E.3.10), the differences in the behavior of the estimators were not as easy to discern. The particular bias of $\hat{\lambda}_1$ and $\hat{\lambda}_3$ was even more apparent, especially at the smaller subsample sizes. As $|\gamma|$ increased, so did the expressions for the asymptotic variances. At each of the subsample sizes, in both types of correlation, $\hat{\lambda}_3$ had the highest estimated variance. The variance of $\hat{\lambda}_3$ was significantly different than that of $\hat{\lambda}_2$ at all levels of significance and subsample sizes up to N = 500. However, we could not reject that the asymptotic variances of $\hat{\lambda}_1$, $\hat{\lambda}_2$ and $\hat{\lambda}_3$ were the same at each of the two levels of correlation.

TABLE III.E.3.5

SIMTBED Summary Statistics for Estimating λ by $\hat{\lambda}_1$ in the BELAR(1) Process with α=.5 and γ=.63662

SUMMARY STATISTICS (MEAN/STD) 5 REPETITIONS

SUBSAMPLE SIZE	25	50	75	100	125	175	250	500
MEAN	0.9478 0.3347E-02	0.9687 0.3894E-02	0.9764 0.3232E-02	0.9782 0.4809E-02	0.9846 0.3164E-02	0.9901 0.3088E-02	0.9925 0.4062E-02	0.9990 0.4749E-02
STD	0.3278 0.3681E-02	0.2458 0.3663E-02	0.2010 0.2204E-02	0.1882 0.2748E-02	0.1648 0.1611E-02	0.1429 0.1675E-02	0.1198 0.3628E-02	0.8573E-01 0.1010E-02
SKEWNESS	1.183 0.4009E-01	0.9325 0.7088E-01	0.8177 0.8252E-01	0.8334 0.9593E-01	0.6410 0.3272E-01	0.5783 0.5749E-01	0.5009 0.8550E-01	0.2104 0.1397
KURTOSIS	2.682 0.4683	1.652 0.3693	1.520 0.4583	1.688 0.6930	0.8507 0.9627E-01	0.5417 0.1780	0.3648 0.2042	0.5342E-01 0.1323
SER.COR.	-0.1230E-01 0.8992E-02	0.8528E-02 0.2468E-01	0.5411E-02 0.1966E-01	0.4915E-02 0.1645E-01	-0.4704E-02 0.1661E-01	-0.2014E-01 0.2139E-01	-0.8800E-01 0.4084E-01	-0.2432E-01 0.1530E-01
QUANTILES								
0.010	0.4137 0.6551E-02	0.5375 0.3283E-02	0.5913 0.4658E-02	0.6164 0.8189E-02	0.6521 0.8685E-02	0.7102 0.6578E-02	0.7591 0.1110E-01	0.8074 0.1240E-01
0.025	0.4717 0.3432E-02	0.5873 0.3164E-02	0.6451 0.8203E-02	0.6691 0.7931E-02	0.7141 0.9089E-02	0.7485 0.4085E-02	0.7815 0.9280E-02	0.8317 0.5580E-02
0.050	0.5226 0.3514E-02	0.6303 0.3641E-02	0.6942 0.4653E-02	0.7168 0.7123E-02	0.7463 0.8074E-02	0.7768 0.4411E-02	0.8101 0.8549E-02	0.8601 0.4844E-02
0.100	0.5876 0.3142E-02	0.6884 0.5286E-02	0.7416 0.5268E-02	0.7614 0.4565E-02	0.7873 0.2701E-02	0.8156 0.3057E-02	0.8507 0.5022E-02	0.8854 0.5844E-02
0.250	0.7182 0.3390E-02	0.7938 0.3528E-02	0.8388 0.2025E-02	0.8444 0.6839E-02	0.8655 0.3275E-02	0.8900 0.2290E-02	0.9055 0.3411E-02	0.9360 0.3634E-02
0.500	0.8956 0.4958E-02	0.9380 0.4268E-02	0.9537 0.4137E-02	0.9586 0.4704E-02	0.9714 0.3187E-02	0.9773 0.3088E-02	0.9799 0.5437E-02	0.9994 0.7871E-02
0.750	1.119 0.4689E-02	1.105 0.7226E-02	1.088 0.3955E-02	1.091 0.1741E-02	1.085 0.5786E-02	1.075 0.7549E-02	1.068 0.3712E-02	1.053 0.6059E-02
0.900	1.365 0.8972E-02	1.288 0.6790E-02	1.238 0.8035E-02	1.229 0.5661E-02	1.197 0.6909E-02	1.180 0.7931E-02	1.158 0.1064E-01	1.104 0.4994E-02
0.950	1.561 0.1186E-01	1.409 0.1239E-01	1.341 0.7392E-02	1.312 0.7357E-02	1.274 0.5331E-02	1.246 0.9817E-02	1.215 0.1837E-01	1.160 0.1219E-01
0.975	1.740 0.3018E-01	1.560 0.2453E-01	1.445 0.6318E-02	1.400 0.6850E-02	1.358 0.1103E-01	1.302 0.1222E-01	1.264 0.2076E-01	1.194 0.1181E-01
0.990	1.995 0.2096E-01	1.743 0.2591E-01	1.574 0.2362E-01	1.529 0.2953E-01	1.478 0.1293E-01	1.404 0.1581E-01	1.325 0.1622E-01	1.228 0.1052E-01

MEAN OF REGRESSION ON AVERAGES - COEFFICIENTS:	1.00564 0.735670E-02	-3.55532 1.52042	119.179 99.4988	-1659.90 1692.71
STD DEV OF REGRESSION - COEFFICIENTS:	0.111250E-01 0.679522E-03	3.08338 0.103674	207.848 6.38264	3452.38 170.573
REGRESSION ON VARIANCE - COEFFICIENTS:	3.53304 0.791226	10.1706 23.1686	-166.323 203.115	471.800 530.823

ESTIMATOR: SQRT(.5*SUM((X-MMU)**2)) LMDA=1.

TABLE III.E.3.6

SIMTBED Summary Statistics for Estimating λ by $\hat{\lambda}_2$ in the BELAR(1) Process with α=.5 and γ=.63662

SUMMARY STATISTICS (MEAN/STD) 5 REPETITIONS

SUBSAMPLE SIZE	25	50	75	100	125	175	250	500
MEAN	1.001 / 0.3888E-02	1.001 / 0.3290E-02	0.9983 / 0.3134E-02	0.9937 / 0.4166E-02	0.9970 / 0.3588E-02	0.9986 / 0.2451E-02	0.9995 / 0.3439E-02	1.004 / 0.4612E-02
STD	0.3408 / 0.3365E-02	0.2435 / 0.2328E-02	0.1946 / 0.1848E-02	0.1755 / 0.1280E-02	0.1526 / 0.1928E-02	0.1306 / 0.2060E-02	0.1107 / 0.2659E-02	0.7863E-01 / 0.1120E-02
SKEWNESS	1.136 / 0.4797E-01	0.8176 / 0.6450E-01	0.6721 / 0.9923E-01	0.5421 / 0.3311E-01	0.4607 / 0.3447E-01	0.4026 / 0.7972E-01	0.3126 / 0.1047	0.2683 / 0.1806
KURTOSIS	2.466 / 0.4035	1.301 / 0.3851	1.221 / 0.5384	0.5076 / 0.2540	0.3255 / 0.1576	0.1911 / 0.1759	-0.5764E-03 / 0.1359	0.4138 / 0.2963
SER.COR.	-0.7745E-02 / 0.1156E-01	0.7909E-02 / 0.2234E-01	0.1079E-01 / 0.2421E-01	0.1185E-01 / 0.2058E-01	-0.1101E-01 / 0.1781E-01	-0.1243E-01 / 0.1616E-01	-0.8359E-01 / 0.3831E-01	-0.6251E-01 / 0.3438E-01
QUANTILES								
0.010	0.4439 / 0.6104E-02	0.5634 / 0.6257E-02	0.6156 / 0.6280E-02	0.6501 / 0.1238E-01	0.6851 / 0.7392E-02	0.7172 / 0.1162E-01	0.7730 / 0.1273E-01	0.8242 / 0.9308E-02
0.025	0.4991 / 0.3390E-02	0.6131 / 0.3232E-02	0.6701 / 0.5707E-02	0.7036 / 0.1031E-01	0.7304 / 0.6991E-02	0.7714 / 0.2674E-02	0.7974 / 0.6407E-02	0.8500 / 0.4951E-02
0.050	0.5550 / 0.2710E-02	0.6575 / 0.2922E-02	0.7116 / 0.4617E-02	0.7386 / 0.7113E-02	0.7702 / 0.6773E-02	0.7967 / 0.2584E-02	0.8285 / 0.7076E-02	0.8771 / 0.4367E-02
0.100	0.6220 / 0.1827E-02	0.7190 / 0.3362E-02	0.7657 / 0.4502E-02	0.7808 / 0.6478E-02	0.8123 / 0.3209E-02	0.8344 / 0.1346E-02	0.8627 / 0.6776E-02	0.9037 / 0.5181E-02
0.250	0.7593 / 0.4507E-02	0.8261 / 0.3528E-02	0.8651 / 0.3654E-02	0.8666 / 0.6762E-02	0.8877 / 0.2227E-02	0.9105 / 0.2727E-02	0.9168 / 0.2471E-02	0.9455 / 0.5158E-02
0.500	0.9464 / 0.6815E-02	0.9734 / 0.4994E-02	0.9812 / 0.2565E-02	0.9776 / 0.2994E-02	0.9883 / 0.2269E-02	0.9850 / 0.2556E-02	0.9935 / 0.5661E-02	1.007 / 0.4003E-02
0.750	1.176 / 0.3770E-02	1.143 / 0.5406E-02	1.112 / 0.6964E-02	1.108 / 0.4632E-02	1.094 / 0.6310E-02	1.081 / 0.6467E-02	1.072 / 0.3615E-02	1.054 / 0.7937E-02
0.900	1.447 / 0.1086E-01	1.312 / 0.8074E-02	1.251 / 0.7849E-02	1.224 / 0.6773E-02	1.195 / 0.9722E-02	1.173 / 0.4606E-02	1.149 / 0.6578E-02	1.094 / 0.7015E-02
0.950	1.634 / 0.1031E-01	1.457 / 0.1551E-01	1.343 / 0.9443E-02	1.300 / 0.5665E-02	1.266 / 0.9003E-02	1.239 / 0.8327E-02	1.199 / 0.1482E-01	1.138 / 0.1200E-01
0.975	1.824 / 0.1098E-01	1.555 / 0.1588E-01	1.433 / 0.9978E-02	1.384 / 0.9695E-02	1.333 / 0.2248E-02	1.277 / 0.1036E-01	1.239 / 0.1464E-01	1.184 / 0.9625E-02
0.990	2.036 / 0.3209E-01	1.715 / 0.4160E-02	1.557 / 0.2606E-01	1.466 / 0.1080E-01	1.416 / 0.2077E-01	1.358 / 0.1876E-01	1.280 / 0.1540E-01	1.244 / 0.2124E-01

MEAN OF REGRESSION ON AVERAGES - COEFFICIENTS:	1.00958 / 0.692602E-02	-2.76952 / 1.44685	178.488 / 93.8504	-2848.81 / 1538.87
STD DEV OF REGRESSION - COEFFICIENTS:	0.987365E-02 / 0.698229E-03	2.77494 / 0.221736	191.357 / 16.7253	3218.48 / 305.642
REGRESSION ON VARIANCE - COEFFICIENTS:	3.56729 / 0.430405	-13.6822 / 13.0028	98.4361 / 116.080	-232.832 / 305.336

ESTIMATOR: SAMPLE MEAN ABS DEV; LMDA=1.0

TABLE III.E.3.7

SIMTBED Summary Statistics for Estimating λ by $\hat{\lambda}_3$ in the BELAR(1) Process with α=.5 and γ=.63662

SUMMARY STATISTICS (MEAN/STD) 5 REPETITIONS

SUBSAMPLE SIZE	25	50	75	100	125	175	250	500
MEAN	1.078 / 0.5631E-02	1.043 / 0.2745E-02	1.027 / 0.3937E-02	1.010 / 0.3232E-02	1.012 / 0.4729E-02	1.007 / 0.4591E-02	1.009 / 0.2301E-02	1.009 / 0.3973E-02
STD	0.4561 / 0.5623E-02	0.3070 / 0.1952E-02	0.2500 / 0.3142E-02	0.2098 / 0.3378E-02	0.1856 / 0.1857E-02	0.1583 / 0.3268E-02	0.1374 / 0.2250E-02	0.9438E-01 / 0.2394E-02
SKEWNESS	1.367 / 0.7416E-01	0.8283 / 0.4437E-01	0.8833 / 0.2339	0.5379 / 0.6569E-01	0.5321 / 0.4230E-01	0.4732 / 0.1084	0.3830 / 0.6686E-01	0.4081 / 0.1167
KURTOSIS	3.668 / 0.5684	1.618 / 0.2496	3.072 / 2.182	0.4043 / 0.2548	0.4181 / 0.1538	0.4186 / 0.3963	0.1145 / 0.1806	0.5837 / 0.3046
SER.COR.	-0.6971E-02 / 0.1084E-01	-0.8897E-02 / 0.9641E-02	0.1644E-01 / 0.1705E-01	0.2105E-01 / 0.1893E-01	-0.1989E-01 / 0.1767E-01	0.1726E-02 / 0.6002E-02	-0.3326E-01 / 0.2540E-01	-0.1056 / 0.4614E-01
QUANTILES								
0.010	0.3704 / 0.4151E-02	0.4926 / 0.2986E-02	0.5590 / 0.6905E-02	0.6248 / 0.1123E-01	0.6336 / 0.5060E-02	0.6825 / 0.7318E-02	0.7280 / 0.1289E-01	0.7867 / 0.9726E-02
0.025	0.4366 / 0.2273E-02	0.5593 / 0.4878E-02	0.6201 / 0.7705E-02	0.6575 / 0.8929E-02	0.6973 / 0.5116E-02	0.7237 / 0.5032E-02	0.7522 / 0.9143E-02	0.8285 / 0.4834E-02
0.050	0.5051 / 0.3383E-02	0.6262 / 0.8666E-02	0.6724 / 0.4032E-02	0.7012 / 0.7583E-02	0.7382 / 0.9339E-02	0.7659 / 0.5855E-02	0.7888 / 0.6298E-02	0.8635 / 0.4160E-02
0.100	0.5864 / 0.3376E-02	0.6928 / 0.6238E-02	0.7274 / 0.6095E-02	0.7556 / 0.4486E-02	0.7844 / 0.6822E-02	0.8124 / 0.3088E-02	0.8397 / 0.6748E-02	0.8944 / 0.2762E-02
0.250	0.7589 / 0.5056E-02	0.8251 / 0.5450E-02	0.8538 / 0.4931E-02	0.8571 / 0.6008E-02	0.8814 / 0.3991E-02	0.8959 / 0.6169E-02	0.9154 / 0.2994E-02	0.9410 / 0.4643E-02
0.500	0.9972 / 0.7864E-02	1.004 / 0.4411E-02	1.004 / 0.5218E-02	0.9935 / 0.5384E-02	0.9945 / 0.3480E-02	0.9905 / 0.4240E-02	1.001 / 0.3820E-02	1.002 / 0.5648E-02
0.750	1.302 / 0.4373E-02	1.215 / 0.5437E-02	1.172 / 0.6103E-02	1.140 / 0.4032E-02	1.132 / 0.4844E-02	1.105 / 0.4459E-02	1.095 / 0.3528E-02	1.067 / 0.6052E-02
0.900	1.663 / 0.1101E-01	1.447 / 0.5892E-02	1.349 / 0.5195E-02	1.297 / 0.1029E-01	1.259 / 0.8159E-02	1.217 / 0.1106E-01	1.191 / 0.6602E-02	1.134 / 0.6540E-02
0.950	1.919 / 0.1790E-01	1.602 / 0.6923E-02	1.474 / 0.8249E-02	1.383 / 0.1350E-01	1.343 / 0.9802E-02	1.292 / 0.1064E-01	1.253 / 0.9645E-02	1.167 / 0.6496E-02
0.975	2.202 / 0.2278E-01	1.757 / 0.6359E-02	1.598 / 0.1680E-01	1.467 / 0.1065E-01	1.417 / 0.9658E-02	1.352 / 0.4927E-02	1.315 / 0.4251E-02	1.213 / 0.1525E-01
0.990	2.571 / 0.3430E-01	1.971 / 0.1969E-01	1.728 / 0.1531E-01	1.605 / 0.2268E-01	1.518 / 0.1658E-01	1.439 / 0.2956E-01	1.392 / 0.1042E-01	1.314 / 0.2811E-01

MEAN OF REGRESSION ON AVERAGES - COEFFICIENTS:	1.01182 / 0.642234E-02	-2.03469 / 1.72452	273.094 / 121.815	-4520.70 / 2115.67
STD DEV OF REGRESSION - COEFFICIENTS:	0.118072E-01 / 0.155488E-02	3.12848 / 0.489336	213.660 / 36.2782	3584.50 / 639.324
REGRESSION ON VARIANCE - COEFFICIENTS:	5.43751 / 0.578759	-28.2587 / 14.0187	237.003 / 111.528	-507.952 / 290.930

ESTIMATOR: SAMPLE MEDIAN ABS DEV; LMDA=1.0

*** WIDEST Y VALUES FOUND: YMIN=0.1882 YMAX= 4.894

TABLE III.E.3.8

SIMTBED Summary Statistics for Estimating λ by $\hat{\lambda}_1$ in the BELAR(1) Process with α=.844 and γ=-.89986

SUMMARY STATISTICS (MEAN/STD) 10 REPETITIONS

SUBSAMPLE SIZE	25	50	75	100	125	175	250	500
MEAN	0.9659 0.1768E-02	0.9162 0.1741E-02	0.9346 0.2733E-02	0.9403 0.3817E-02	0.9531 0.3684E-02	0.9649 0.2962E-02	0.9760 0.4195E-02	0.9851 0.3312E-02
STD	0.4896 0.3321E-02	0.4037 0.3875E-02	0.3499 0.3906E-02	0.3192 0.4198E-02	0.2921 0.3084E-02	0.2517 0.5182E-02	0.2220 0.3500E-02	0.1592 0.3017E-02
SKEWNESS	1.708 0.3993E-01	1.326 0.7607E-01	1.229 0.5790E-01	1.243 0.7056E-01	1.095 0.4579E-01	0.9764 0.7580E-01	0.7820 0.6649E-01	0.6045 0.4423E-01
KURTOSIS	4.933 0.2914	3.493 0.4838	2.816 0.6028	2.939 0.4679	2.064 0.2558	1.895 0.5026	1.129 0.2619	0.5740 0.1242
SER.COR.	0.5375E-03 0.5248E-02	-0.6024E-03 0.6357E-02	0.8106E-03 0.9839E-02	-0.3326E-03 0.8359E-02	-0.4255E-03 0.9663E-02	-0.1402E-01 0.1136E-01	-0.2824E-03 0.1331E-01	-0.3432E-01 0.1411E-01
QUANTILES								
0.010	0.2101 0.2072E-02	0.3147 0.2771E-02	0.3819 0.3731E-02	0.4303 0.4920E-02	0.4702 0.6029E-02	0.5194 0.6971E-02	0.5702 0.4132E-02	0.6620 0.8971E-02
0.025	0.2612 0.2173E-02	0.3666 0.3375E-02	0.4363 0.3234E-02	0.4873 0.3673E-02	0.5219 0.3524E-02	0.5753 0.4528E-02	0.6175 0.4810E-02	0.7106 0.5797E-02
0.050	0.3088 0.1917E-02	0.4172 0.1598E-02	0.4892 0.2046E-02	0.5351 0.3110E-02	0.5660 0.4987E-02	0.6189 0.2783E-02	0.6612 0.5556E-02	0.7511 0.7168E-02
0.100	0.3795 0.1437E-02	0.4882 0.2248E-02	0.5557 0.1881E-02	0.6004 0.2336E-02	0.6293 0.4076E-02	0.6818 0.3052E-02	0.7137 0.3716E-02	0.7989 0.5029E-02
0.250	0.5254 0.1830E-02	0.6303 0.2509E-02	0.6856 0.1940E-02	0.7246 0.1801E-02	0.7460 0.3129E-02	0.7902 0.2626E-02	0.8230 0.3823E-02	0.8734 0.3071E-02
0.500	0.7544 0.2205E-02	0.8406 0.2046E-02	0.8696 0.2586E-02	0.8971 0.3199E-02	0.9063 0.4263E-02	0.9285 0.3520E-02	0.9494 0.4920E-02	0.9689 0.2670E-02
0.750	1.078 0.3194E-02	1.113 0.3470E-02	1.117 0.6284E-02	1.110 0.5778E-02	1.109 0.5976E-02	1.102 0.4564E-02	1.101 0.6626E-02	1.080 0.3989E-02
0.900	1.487 0.5661E-02	1.436 0.4907E-02	1.395 0.7179E-02	1.355 0.8438E-02	1.325 0.7203E-02	1.296 0.8173E-02	1.269 0.7856E-02	1.200 0.8631E-02
0.950	1.815 0.9417E-02	1.679 0.9665E-02	1.591 0.1382E-01	1.540 0.1535E-01	1.495 0.9604E-02	1.428 0.1205E-01	1.388 0.1067E-01	1.272 0.8173E-02
0.975	2.135 0.1241E-01	1.912 0.1432E-01	1.786 0.1556E-01	1.733 0.1552E-01	1.662 0.1720E-01	1.552 0.2325E-01	1.492 0.1263E-01	1.353 0.2122E-01
0.990	2.574 0.2032E-01	2.266 0.4704E-01	2.038 0.1545E-01	2.023 0.4142E-01	1.903 0.2468E-01	1.735 0.3453E-01	1.637 0.3717E-01	1.454 0.1955E-01

MEAN OF REGRESSION ON AVERAGES - COEFFICIENTS:	0.998311 0.320766E-02	-6.63699 0.991931	170.407 69.0520	-2179.42 1166.40
STD DEV OF REGRESSION - COEFFICIENTS:	0.152742E-01 0.116903E-02	3.52568 0.322466	210.881 20.6383	3347.29 340.527
REGRESSION ON VARIANCE - COEFFICIENTS:	15.0734 1.99159	-43.6055 57.2045	-111.968 491.153	525.446 1274.95

ESTIMATOR: SQRT(.5*SUM((X-MMU)**2)) LMDA=1.

TABLE III.E.3.9

SIMTBED Summary Statistics for Estimating λ by $\hat{\lambda}_2$ in the BELAR(1) Process with α=.844 and γ=-.89986

SUMMARY STATISTICS (MEAN/STD) 10 REPETITIONS

SUBSAMPLE SIZE	25	50	75	100	125	175	250	500
MEAN	0.9930 / 0.2041E-02	1.001 / 0.1624E-02	0.9991 / 0.2806E-02	1.001 / 0.3566E-02	0.9975 / 0.4254E-02	0.9973 / 0.3328E-02	1.001 / 0.3626E-02	0.9977 / 0.3485E-02
STD	0.6024 / 0.3965E-02	0.4563 / 0.4940E-02	0.3796 / 0.3890E-02	0.3342 / 0.3342E-02	0.2995 / 0.3508E-02	0.2510 / 0.4483E-02	0.2177 / 0.3097E-02	0.1495 / 0.2286E-02
SKEWNESS	1.913 / 0.4143E-01	1.477 / 0.9506E-01	1.241 / 0.4675E-01	1.184 / 0.7861E-01	1.008 / 0.4395E-01	0.8167 / 0.4206E-01	0.6874 / 0.6714E-01	0.4148 / 0.3164E-01
KURTOSIS	6.241 / 0.3243	4.047 / 0.6800	2.823 / 0.4503	2.740 / 0.5828	1.762 / 0.2607	1.142 / 0.1499	0.7651 / 0.2352	0.1318 / 0.9039E-01
SER.COR.	0.4574E-02 / 0.4660E-02	-0.3538E-02 / 0.6471E-02	-0.2623E-02 / 0.9240E-02	0.2434E-03 / 0.9090E-02	-0.8504E-02 / 0.1127E-01	-0.1462E-01 / 0.1208E-01	-0.2014E-01 / 0.1311E-01	-0.5852E-01 / 0.1819E-01
QUANTILES								
0.010	0.2251 / 0.3272E-02	0.3353 / 0.1688E-02	0.4051 / 0.4178E-02	0.4542 / 0.6568E-02	0.4925 / 0.6466E-02	0.5443 / 0.5373E-02	0.5896 / 0.8902E-02	0.6866 / 0.8402E-02
0.025	0.2795 / 0.2295E-02	0.3897 / 0.3578E-02	0.4631 / 0.3489E-02	0.5088 / 0.3901E-02	0.5442 / 0.5378E-02	0.6011 / 0.4362E-02	0.6409 / 0.5112E-02	0.7300 / 0.6746E-02
0.050	0.3355 / 0.1904E-02	0.4475 / 0.2606E-02	0.5165 / 0.3674E-02	0.5650 / 0.3661E-02	0.5922 / 0.4350E-02	0.6487 / 0.4121E-02	0.6883 / 0.4974E-02	0.7697 / 0.6548E-02
0.100	0.4144 / 0.1277E-02	0.5228 / 0.2370E-02	0.5881 / 0.2502E-02	0.6280 / 0.1605E-02	0.6615 / 0.4064E-02	0.7065 / 0.4228E-02	0.7443 / 0.6042E-02	0.8158 / 0.3384E-02
0.250	0.5822 / 0.2505E-02	0.6797 / 0.2374E-02	0.7301 / 0.2208E-02	0.7652 / 0.2174E-02	0.7847 / 0.4157E-02	0.8198 / 0.3098E-02	0.8460 / 0.4243E-02	0.8904 / 0.3734E-02
0.500	0.8477 / 0.2999E-02	0.9127 / 0.2455E-02	0.9312 / 0.2996E-02	0.9465 / 0.3252E-02	0.9552 / 0.3972E-02	0.9652 / 0.3830E-02	0.9777 / 0.5101E-02	0.9851 / 0.3948E-02
0.750	1.237 / 0.3778E-02	1.214 / 0.3600E-02	1.194 / 0.5099E-02	1.175 / 0.6777E-02	1.160 / 0.5582E-02	1.140 / 0.4814E-02	1.127 / 0.4804E-02	1.089 / 0.6221E-02
0.900	1.748 / 0.5192E-02	1.586 / 0.7250E-02	1.498 / 0.9068E-02	1.428 / 0.7942E-02	1.386 / 0.9425E-02	1.333 / 0.9380E-02	1.291 / 0.7320E-02	1.201 / 0.6665E-02
0.950	2.154 / 0.1162E-01	1.862 / 0.8889E-02	1.714 / 0.8715E-02	1.620 / 0.1082E-01	1.542 / 0.1170E-01	1.456 / 0.1407E-01	1.400 / 0.7215E-02	1.269 / 0.7626E-02
0.975	2.580 / 0.1647E-01	2.142 / 0.1800E-01	1.914 / 0.1500E-01	1.832 / 0.1379E-01	1.731 / 0.1736E-01	1.589 / 0.2128E-01	1.525 / 0.2311E-01	1.329 / 0.1009E-01
0.990	3.131 / 0.2834E-01	2.532 / 0.4950E-01	2.200 / 0.1935E-01	2.093 / 0.3846E-01	1.952 / 0.1644E-01	1.731 / 0.2798E-01	1.632 / 0.2499E-01	1.402 / 0.1636E-01

MEAN OF REGRESSION ON AVERAGES - COEFFICIENTS:	0.998834 / 0.361607E-02	-0.193866 / 1.15998	29.6152 / 80.2653	-708.950 / 1347.23
STD DEV OF REGRESSION - COEFFICIENTS:	0.147114E-01 / 0.135324E-02	3.63334 / 0.360922	229.778 / 21.3799	3741.57 / 343.306
REGRESSION ON VARIANCE - COEFFICIENTS:	11.0067 / 2.13113	18.1352 / 62.5665	-205.042 / 548.121	330.485 / 1439.15

ESTIMATOR: SAMPLE MEAN ABS DEV; LMDA=1.0

*** WIDEST Y VALUES FOUND: YMIN=0.3846E-01 YMAX= 12.24

TABLE III.E.3.10

SIMTBED Summary Statistics for Estimating λ by $\hat{\lambda}_3$ in the BELAR(1) Process with α=.844 and γ=-.89986

SUMMARY STATISTICS (MEAN/STD) 10 REPETITIONS

SUBSAMPLE SIZE	25	50	75	100	125	175	250	500
MEAN	1.226 / 0.2668E-02	1.143 / 0.2526E-02	1.099 / 0.3723E-02	1.076 / 0.4382E-02	1.061 / 0.5845E-02	1.037 / 0.4643E-02	1.035 / 0.3740E-02	1.012 / 0.4860E-02
STD	0.9303 / 0.6881E-02	0.6474 / 0.8429E-02	0.5150 / 0.3641E-02	0.4341 / 0.5014E-02	0.3802 / 0.4354E-02	0.3076 / 0.5669E-02	0.2622 / 0.4070E-02	0.1732 / 0.1586E-02
SKEWNESS	2.301 / 0.6786E-01	1.849 / 0.1115	1.507 / 0.3756E-01	1.465 / 0.1629	1.179 / 0.1037	0.8759 / 0.6567E-01	0.7977 / 0.6257E-01	0.4464 / 0.7947E-01
KURTOSIS	9.043 / 0.7120	6.469 / 0.9986	3.850 / 0.2809	4.914 / 1.807	2.699 / 0.6134	1.172 / 0.2166	0.9677 / 0.3346	0.2372 / 0.1645
SER.COR.	0.4268E-03 / 0.5101E-02	0.2205E-02 / 0.6719E-02	-0.6409E-02 / 0.8728E-02	0.2884E-02 / 0.9203E-02	-0.5370E-02 / 0.1278E-01	-0.1609E-01 / 0.1276E-01	-0.1469E-01 / 0.1393E-01	-0.6675E-01 / 0.1823E-01
QUANTILES								
0.010	0.1760 / 0.2755E-02	0.2887 / 0.5995E-02	0.3549 / 0.5294E-02	0.4093 / 0.4733E-02	0.4511 / 0.1026E-01	0.4953 / 0.6239E-02	0.5532 / 0.6473E-02	0.6583 / 0.1216E-01
0.025	0.2356 / 0.2785E-02	0.3482 / 0.4377E-02	0.4157 / 0.5340E-02	0.4751 / 0.3193E-02	0.4996 / 0.6865E-02	0.5673 / 0.3010E-02	0.6108 / 0.5125E-02	0.7041 / 0.4641E-02
0.050	0.2991 / 0.2968E-02	0.4118 / 0.3439E-02	0.4806 / 0.4070E-02	0.5358 / 0.3163E-02	0.5644 / 0.4858E-02	0.6230 / 0.3288E-02	0.6661 / 0.6405E-02	0.7418 / 0.3085E-02
0.100	0.3915 / 0.2511E-02	0.5048 / 0.2793E-02	0.5645 / 0.4569E-02	0.6132 / 0.2768E-02	0.6434 / 0.4616E-02	0.6820 / 0.4228E-02	0.7380 / 0.3825E-02	0.8006 / 0.4143E-02
0.250	0.6091 / 0.3052E-02	0.6952 / 0.2879E-02	0.7395 / 0.3313E-02	0.7717 / 0.2281E-02	0.7821 / 0.6037E-02	0.8155 / 0.6016E-02	0.8487 / 0.4420E-02	0.8897 / 0.3846E-02
0.500	0.9701 / 0.4293E-02	0.9970 / 0.3660E-02	0.9920 / 0.4574E-02	1.001 / 0.7555E-02	1.000 / 0.4727E-02	0.9928 / 0.6226E-02	0.9992 / 0.3158E-02	1.001 / 0.6420E-02
0.750	1.556 / 0.6023E-02	1.419 / 0.4855E-02	1.335 / 0.4511E-02	1.291 / 0.7984E-02	1.258 / 0.6754E-02	1.212 / 0.8374E-02	1.182 / 0.3628E-02	1.121 / 0.6466E-02
0.900	2.343 / 0.6038E-02	1.953 / 0.7834E-02	1.766 / 0.7423E-02	1.627 / 0.9942E-02	1.556 / 0.1290E-01	1.449 / 0.1231E-01	1.388 / 0.8132E-02	1.246 / 0.4857E-02
0.950	3.012 / 0.1520E-01	2.362 / 0.1125E-01	2.075 / 0.1262E-01	1.866 / 0.1143E-01	1.779 / 0.1397E-01	1.613 / 0.1865E-01	1.529 / 0.1080E-01	1.323 / 0.8657E-02
0.975	3.729 / 0.2588E-01	2.827 / 0.2578E-01	2.395 / 0.1543E-01	2.143 / 0.1924E-01	2.002 / 0.1551E-01	1.763 / 0.2185E-01	1.641 / 0.1975E-01	1.403 / 0.1296E-01
0.990	4.719 / 0.5287E-01	3.407 / 0.7377E-01	2.878 / 0.3106E-01	2.529 / 0.4828E-01	2.274 / 0.3643E-01	1.957 / 0.3643E-01	1.837 / 0.3995E-01	1.482 / 0.1826E-01

MEAN OF REGRESSION ON AVERAGES - COEFFICIENTS:	0.996838 / 0.535875E-02	0.10085 / 1.53969	-23.5816 / 105.041	-942.467 / 1753.68
STD DEV OF REGRESSION - COEFFICIENTS:	0.179159E-01 / 0.190478E-02	5.16999 / 0.405652	348.422 / 20.4377	5827.03 / 328.305
REGRESSION ON VARIANCE - COEFFICIENTS:	13.2108 / 3.99996	28.7581 / 118.852	480.708 / 1052.46	-2068.37 / 2789.21

ESTIMATOR: SAMPLE MEDIAN ABS DEV; LMDA=1.0

4. Least Squares Estimation of Serial Correlation

In this section, it is assumed, unless otherwise stated, that X_n has a standard Laplace ($\mu = 0, \lambda = 1$) distribution. If not, standardize X_n by

$$X'_n = \frac{X_n - \hat{\mu}}{\hat{\lambda}}, \qquad \text{(III.E.4.1)}$$

where $\hat{\mu}$ and $\hat{\lambda}$ will be specified from those estimators already discussed in III.E.2 and III.E.3.

The least squares estimator of the lag-1 serial corrlation, $\hat{\gamma}_{LS}$, is derived. First, we show that the BELAR(1) process is an RCA(1) process of Nicholls and Quinn [Ref. 16]. Then, we define the linearized residual in the BELAR(1) process and state some of its properties. From these properties and some results from Nicholls and Quinn for RCA processes, we derive the asymptotic properties of $\hat{\gamma}_{LS}$. The properties of $\hat{\gamma}_{LS}$ are observed also in the simulation results for selected values of γ. Finally, the joint least squares estimator of location and serial correlation are derived for the BELAR(1) process.

Rewriting (III.D.1.1) by adding and subtracting γX_{n-1}, we have

$$X_n = \gamma X_{n-1} + \{A_n^{1/2}(\alpha,1-\alpha) - \gamma\}X_{n-1} + \varepsilon_n, \qquad \text{(III.E.4.2)}$$

where $\gamma = E\{A_n^{1/2}(\alpha,1-\alpha)\}$ as given by (III.C.2.3) for $\ell = 1$; $\{A_n^{1/2}(\alpha,1-\alpha) - \gamma\}$ is an i.i.d. process stochastically independent of the i.i.d. $\{\varepsilon_n\}$. The variance of the random coefficient is $(\alpha - \gamma^2)$

for all n. As can be seen from (III.C.2.5) and the fact that $0 < \alpha < 1$, if we know α, then we also know $|\gamma|$ and vice-versa. That is, in the BELAR(1) process, there is only one independent parameter to estimate for the correlation. Now, we recognize (III.E.4.2) immediately as an RCA(1) process of Nicholls and Quinn [Ref. 16]. Since $\{\varepsilon_n\}$ and $\{A_n^{1/2}(\alpha,1-\alpha) - \gamma\}$ are each identically distributed as well as being serially independent and independent of each other, we have by theorem 2.7 [Ref. 16] that $\{X_n\}$ is the unique strictly stationary and ergodic solution to (III.E.4.2).

There are two ways to look at the linearized residual in the BELAR(1) process just as described in Chapter II for the NLAR(1) model:

$$R_n = \{A_n^{1/2}(\alpha,1-\alpha) - \gamma\}X_{n-1} + \varepsilon_n, \qquad \text{(III.E.4.3)}$$

or

$$R_n = X_n - \gamma X_{n-1}. \qquad \text{(III.E.4.4)}$$

From (III.E.4.4), we see that since $\{X_n\}$ is strictly stationary, so is $\{R_n\}$. Also, we see $E(R_n) = 0$ and $\mathrm{Var}(R_n) = 2(1-\gamma^2)$. Lawrance and Lewis [Ref. 22] proved that the R_n are uncorrelated, but in general, not independent. From (III.E.4.3), we note that for any n, $R_n \neq \varepsilon_n$ unless $\alpha = 0$. Except for when $\alpha = 0$ or 1, $\mathrm{Var}(R_n) > \mathrm{Var}(\varepsilon_n)$. As α increases from zero to one, both $\mathrm{Var}(R_n)$ and $\mathrm{Var}(\varepsilon_n)$ decrease monotonically from two to zero. This is evident from the definition of γ in (III.C.2.5) with $\ell = 1$.

Two other properties of $\{R_n\}$ are obtained from (III.E.4.3) by conditioning on the independent, identically distributed processes $\{\varepsilon_k\}$ and $\{A_k^{1/2}(\alpha,1-\alpha) - \gamma\}$ up to time $k = n - 1$. We have

$$E[R_n|\{\varepsilon_k, A_k^{1/2}(\alpha,1-\alpha) - \gamma\};\quad k = 1,2,\ldots,n-1]$$

$$= x_{n-1}E\{A_n^{1/2}(\alpha\ 1-\alpha) - \gamma\} + E(\varepsilon_n) = 0, \qquad \text{(III.E.4.5)}$$

because $\{A_n^{1/2}(\alpha,1-\alpha) - \gamma\}$ and ε_n are independent of the process through time n-1 and X_{n-1} is a function only of the process through n-1.

$$E[R_n^2|\{\varepsilon_k,\ A_k^{1/2}(\alpha,1-\alpha) - \gamma\};\ k = 1,2,\ldots,n-1]$$

$$= E(\varepsilon_n^2) + x_{n-1}^2E[\{A_n^{1/2}(\alpha,1-\alpha) - \gamma\}^2]$$

$$= 2(1-\alpha) + x_{n-1}^2(\alpha-\gamma^2), \qquad \text{(III.E.4.6)}$$

which is only a function of α or γ^2 alone, since α determines γ^2 and vice-versa.

Now using (III.E.4.4) and a given realization of $\{X_n\}$ of size n, we minimize $\sum_{i=2}^{n} R_i^2$ with respect to γ to obtain the conditional least squares estimate for γ. This is the same procedure as described for the NLAR(1) process. We have

$$\hat{\gamma}_{LS} = \left(\sum_{i=2}^{n} x_i x_{i-1}\right) / \left(\sum_{i=2}^{n} x_{i-1}^2\right). \qquad \text{(III.E.4.7)}$$

Two problems can occur using (III.E.4.7), especially for small sample sizes. For the BELAR(1) process defined by (III.E.4.2), $1 \geq \gamma \geq 0$, and yet it is possible that $\hat{\gamma}_{LS} < 0$ or $|\hat{\gamma}_{LS}| > 1$. If $-1 < \hat{\gamma}_{LS} < 0$, we would estimate that the sample $\{X_n\}$ came from the BELAR(1) process with the negative sign on $A_n^{1/2}(\alpha, 1-\alpha)$. If $|\hat{\gamma}| > 1$, we would estimate γ by +1 or -1.

In order to obtain the "least squares" estimate for α, we solve numerically for $\hat{\alpha}_{LS}$ in

$$|\hat{\gamma}_{LS}| = \frac{2}{\sqrt{\pi}} \frac{\Gamma(\hat{\alpha}_{LS}+1/2)}{\Gamma(\hat{\alpha}_{LS})}, \qquad \text{(III.E.4.8)}$$

for a given $\hat{\gamma}_{LS}$ from (5.7) if $|\hat{\gamma}_{LS}| < 1$.

The estimator $\hat{\gamma}_{LS}$ given by (III.E.4.7) has the following properties which we state as a corollary to Theorm 3.1 [Ref. 16]:

CORROLLARY. For $\{X_n\}$ given by (III.E.4.2); $\{R_n\}$ in (III.E.4.3) and (III.E.4.4), the least squares estimator $\hat{\gamma}_{LS}$ has the following properties:

a) $\hat{\gamma}_{LS} \xrightarrow{a.s.} \gamma$;

b) Since $E(X_n^4) = 24 < \infty$, $(N-1)^{1/2}(\hat{\gamma}_{LS}-\gamma)$ has a distribution which converges to the Normal with a mean of zero and a variance σ_γ^2 given by

$$\sigma_\gamma^2 = 1+5\alpha-6\gamma^2. \qquad \text{(III.E.4.9)}$$

The proof follows from Theorem(3.1). The strict stationarity and ergodic nature of $\{X_n\}$ leads to the almost sure convergence. The results of (III.E.4.5) and (III.E.4.6), together with Billingsley's Martingale Central Limit Theorem provide the results for the asymptotic Normality of $\hat{\gamma}_{LS}$.

A strongly consistent estimator for the variance, σ_γ^2, is also given in [Ref. 16] for the general RCA(1) process. For σ_γ^2 in (III.E.4.9), this estimate becomes

$$\hat{\sigma}_\gamma^2 = \frac{(n-1)}{\sum_{i=2}^{n} X^2_{i-1}} \left\{ \frac{(1-\hat{\alpha}_{LS})}{n} \sum_{i=1}^{n} X_i^2 + \frac{(\hat{\alpha}_{LS}-\hat{\gamma}^2_{LS}) \sum_{i=2}^{n} X^4_{i-1}}{\sum_{i=2}^{n} X^2_{i-1}} \right\}. \qquad \text{(III.E.4.10)}$$

For large n (III.E.4.10) is approximated by

$$\hat{\sigma}_\gamma^2 \approx (1-\hat{\alpha}_{LS}) + \frac{(n-1)(\hat{\alpha}_{LS}-\hat{\gamma}^2_{LS}) \sum_{i=2}^{n} X^4_{i-1}}{\left\{ \sum_{i=2}^{n} X^2_{i-1} \right\}^2}, \qquad \text{(III.E.4.11)}$$

where $\hat{\gamma}_{LS}$ is from (III.E.4.7) and $\hat{\alpha}_{LS}$ (III.E.4.8).

Simulations of the least squares estimator of γ were conducted for selected values of γ in SIMTBED using Type III plans. The results are summarized in Tables III.E.4.1, III.E.4.2 and III.E.4.3. The

TABLE III.E.4.1

SIMTBED Summary Statistics for Estimating γ by the Least Squares Estimator, $\hat{\gamma}_{LS}$, in the BELAR(1) Process with α=.5 and γ=.63662

SUMMARY STATISTICS (MEAN/STD) 10 REPETITIONS

SUBSAMPLE SIZE	25	50	75	100	125	175	250	500
MEAN	0.5755 0.6591E-03	0.6001 0.7987E-03	0.6119 0.4367E-03	0.6154 0.6905E-03	0.6122 0.6429E-03	0.6245 0.9434E-03	0.6277 0.8828E-03	0.6324 0.5823E-03
STD	0.1840 0.5205E-03	0.1331 0.3395E-03	0.1103 0.3326E-03	0.9653E-01 0.4575E-03	0.8667E-01 0.4306E-03	0.7423E-01 0.6401E-03	0.6307E-01 0.5487E-03	0.4547E-01 0.6313E-03
SKEWNESS	-0.6607 0.1287E-01	-0.5823 0.9829E-02	-0.5117 0.1606E-01	-0.4254 0.2344E-01	-0.3741 0.2147E-01	-0.2941 0.1612E-01	-0.2526 0.4000E-01	-0.1606 0.4426E-01
KURTOSIS	0.6565 0.4053E-01	0.4891 0.4835E-01	0.3930 0.5408E-01	0.3182 0.9332E-01	0.1708 0.4118E-01	0.1867 0.9180E-01	0.2183 0.8463E-01	0.2381E-01 0.1315
SER.COR.	-0.2051E-02 0.2708E-02	-0.3298E-02 0.5123E-02	0.9462E-02 0.3646E-02	-0.2395E-02 0.6737E-02	0.8996E-02 0.2391E-02	0.1134E-02 0.6286E-02	-0.1823E-01 0.1453E-01	-0.2826E-01 0.1747E-01
QUANTILES								
0.010	0.5730E-01 0.5465E-02	0.2375 0.1484E-02	0.3092 0.3395E-02	0.3608 0.4025E-02	0.3946 0.3474E-02	0.4350 0.3323E-02	0.4633 0.4808E-02	0.5155 0.4311E-02
0.025	0.1607 0.3108E-02	0.3016 0.1520E-02	0.3681 0.2186E-02	0.4081 0.2341E-02	0.4362 0.2819E-02	0.4697 0.3378E-02	0.4928 0.2893E-02	0.5386 0.2624E-02
0.050	0.2398 0.2209E-02	0.3596 0.1634E-02	0.4143 0.1815E-02	0.4461 0.1931E-02	0.4672 0.1887E-02	0.4991 0.2215E-02	0.5200 0.1722E-02	0.5564 0.1099E-02
0.100	0.3283 0.1165E-02	0.4228 0.1634E-02	0.4638 0.1019E-02	0.4882 0.1326E-02	0.5041 0.8235E-03	0.5274 0.1929E-02	0.5472 0.1209E-02	0.5743 0.9434E-03
0.250	0.4653 0.1019E-02	0.5206 0.1142E-02	0.5435 0.7564E-03	0.5553 0.7279E-03	0.5635 0.7206E-03	0.5767 0.1286E-02	0.5877 0.8674E-03	0.6016 0.1200E-02
0.500	0.5982 0.8855E-03	0.6134 0.7206E-03	0.6208 0.7564E-03	0.6232 0.7279E-03	0.6252 0.5249E-03	0.6277 0.8735E-03	0.6298 0.9490E-03	0.6334 0.9149E-03
0.750	0.7101 0.7634E-03	0.6957 0.8735E-03	0.6891 0.8299E-03	0.6833 0.6828E-03	0.6801 0.8489E-03	0.6761 0.9091E-03	0.6716 0.7132E-03	0.6638 0.9820E-03
0.900	0.7906 0.7351E-03	0.7593 0.8105E-03	0.7445 0.7703E-03	0.7338 0.8974E-03	0.7270 0.1222E-02	0.7173 0.8795E-03	0.7054 0.9378E-03	0.6895 0.1118E-02
0.950	0.8329 0.9601E-03	0.7942 0.9033E-03	0.7747 0.7206E-03	0.7618 0.1294E-02	0.7519 0.9927E-03	0.7401 0.1045E-02	0.7257 0.1174E-02	0.7062 0.1099E-02
0.975	0.8664 0.1094E-02	0.8209 0.1621E-02	0.7993 0.1467E-02	0.7841 0.1527E-02	0.7738 0.6591E-03	0.7600 0.1334E-02	0.7442 0.1346E-02	0.7189 0.1565E-02
0.990	0.9045 0.1499E-02	0.8519 0.2262E-02	0.8275 0.2557E-02	0.8128 0.1128E-02	0.7964 0.1338E-02	0.7840 0.1178E-02	0.7654 0.2056E-02	0.7324 0.2377E-02

MEAN OF REGRESSION ON AVERAGES - COEFFICIENTS:	0.637151 0.909130E-03	-2.48437 0.261786	41.2104 18.5776	-437.273 320.199
STD DEV OF REGRESSION - COEFFICIENTS:	0.330022E-02 0.133123E-03	0.841674 0.512896E-01	54.9135 3.73098	904.088 60.1420
REGRESSION ON VARIANCE - COEFFICIENTS:	1.18420 0.106512	-4.15250 2.77905	19.7294 22.6880	-37.0919 57.5033

ESTIMATOR: LEAST SQUARES USING LINEAR RESIDUAL FROM BELAR(1) RANDOM COEFFICIENT PROCESS; RHO = 0.63662.

*** WIDEST Y VALUES FOUND: YMIN=-.5796 YMAX= 1.478

TABLE III.E.4.2

SIMTBED Summary Statistics for Estimating γ by the Least Squares Estimator, $\hat{\gamma}_{LS}$, in the BELAR(1) Process with α=.2 and γ=.31905

SUMMARY STATISTICS (MEAN/STD)   10 REPETITIONS

SUBSAMPLE SIZE	25	50	75	100	125	175	250	500
MEAN	0.2865 / 0.6896E-03	0.2981 / 0.5217E-03	0.3046 / 0.7104E-03	0.3069 / 0.8628E-03	0.3103 / 0.8036E-03	0.3125 / 0.6418E-03	0.3144 / 0.7823E-03	0.3173 / 0.9149E-03
STD	0.2061 / 0.4028E-03	0.1511 / 0.6531E-03	0.1273 / 0.6319E-03	0.1113 / 0.5280E-03	0.1010 / 0.4118E-03	0.8480E-01 / 0.4205E-03	0.7275E-01 / 0.6495E-03	0.5137E-01 / 0.4636E-03
SKEWNESS	-0.2438 / 0.4415E-02	-0.1756 / 0.6540E-02	-0.1601 / 0.1385E-01	-0.9686E-01 / 0.1337E-01	-0.7876E-01 / 0.3137E-01	-0.5832E-01 / 0.2345E-01	0.5593E-02 / 0.3733E-01	-0.4896E-03 / 0.5015E-01
KURTOSIS	-0.8256E-01 / 0.2154E-01	-0.6936E-01 / 0.2855E-01	-0.2907E-01 / 0.3661E-01	-0.8457E-01 / 0.4972E-01	0.2088E-01 / 0.3032E-01	-0.3836E-02 / 0.3375E-01	0.8996E-01 / 0.8893E-01	0.1876E-01 / 0.5569E-01
SER.COR.	0.8792E-03 / 0.4376E-02	-0.4411E-02 / 0.6249E-01	0.7695E-02 / 0.6055E-02	0.6784E-02 / 0.7943E-02	0.9514E-02 / 0.9825E-02	0.9722E-02 / 0.9093E-02	0.1182E-01 / 0.1161E-01	-0.2645E-01 / 0.1429E-01
QUANTILES								
0.010	-0.2240 / 0.2948E-02	-0.7061E-01 / 0.2954E-02	-0.4048E-02 / 0.3482E-02	0.4176E-01 / 0.3193E-02	0.6708E-01 / 0.3409E-02	0.1115 / 0.2632E-02	0.1392 / 0.3258E-02	0.1951 / 0.2293E-02
0.025	-0.1381 / 0.1633E-02	-0.1111E-01 / 0.1697E-02	0.4813E-01 / 0.2392E-02	0.8188E-01 / 0.2247E-02	0.1079 / 0.1631E-02	0.1416 / 0.1853E-02	0.1691 / 0.2552E-02	0.2165 / 0.1845E-02
0.050	-0.6815E-01 / 0.1290E-02	0.4227E-01 / 0.1106E-02	0.8786E-01 / 0.1638E-02	0.1197 / 0.1894E-02	0.1405 / 0.1650E-02	0.1691 / 0.1388E-02	0.1944 / 0.2185E-02	0.2324 / 0.1190E-02
0.100	0.1279E-01 / 0.1012E-02	0.9929E-01 / 0.1384E-02	0.1375 / 0.1211E-02	0.1615 / 0.1667E-02	0.1807 / 0.1223E-02	0.2037 / 0.1236E-02	0.2203 / 0.1615E-02	0.2509 / 0.1284E-02
0.250	0.1500 / 0.8574E-03	0.1978 / 0.1125E-02	0.2199 / 0.9178E-03	0.2323 / 0.1227E-02	0.2431 / 0.9880E-03	0.2558 / 0.1021E-02	0.2656 / 0.1057E-02	0.2829 / 0.9297E-03
0.500	0.2968 / 0.1060E-02	0.3029 / 0.9247E-03	0.3088 / 0.8195E-03	0.3094 / 0.7365E-03	0.3118 / 0.1260E-02	0.3131 / 0.9146E-03	0.3151 / 0.8015E-03	0.3165 / 0.1499E-02
0.750	0.4340 / 0.9820E-03	0.4049 / 0.9149E-03	0.3930 / 0.9927E-03	0.3844 / 0.9207E-03	0.3796 / 0.1481E-02	0.3705 / 0.8551E-03	0.3641 / 0.1085E-02	0.3524 / 0.1045E-02
0.900	0.5464 / 0.8040E-03	0.4899 / 0.8812E-03	0.4651 / 0.1565E-02	0.4484 / 0.1050E-02	0.4397 / 0.1123E-02	0.4210 / 0.5147E-03	0.4069 / 0.1647E-02	0.3838 / 0.1248E-02
0.950	0.6067 / 0.1481E-02	0.5379 / 0.1169E-02	0.5077 / 0.1571E-02	0.4871 / 0.1456E-02	0.4728 / 0.1434E-02	0.4495 / 0.1827E-02	0.4328 / 0.1691E-02	0.4028 / 0.2258E-02
0.975	0.6554 / 0.1624E-02	0.5768 / 0.1286E-02	0.5451 / 0.1950E-02	0.5205 / 0.9927E-03	0.5044 / 0.1187E-02	0.4761 / 0.2048E-02	0.4542 / 0.1554E-02	0.4187 / 0.2385E-02
0.990	0.7123 / 0.1513E-02	0.6249 / 0.1827E-02	0.5839 / 0.2897E-02	0.5549 / 0.2004E-02	0.5413 / 0.2941E-02	0.5086 / 0.2549E-02	0.4834 / 0.4139E-02	0.4370 / 0.2383E-02

MEAN OF REGRESSION ON AVERAGES - COEFFICIENTS:	0.319807 / 0.770323E-03	-1.34965 / 0.200646	14.3510 / 13.7684	-33.9005 / 234.102
STD DEV OF REGRESSION - COEFFICIENTS:	0.341165E-02 / 0.211696E-03	0.867991 / 0.485546E-01	55.2193 / 2.55757	911.765 / 42.3921
REGRESSION ON VARIANCE - COEFFICIENTS:	1.24507 / 0.132501	3.49087 / 3.49898	-48.2355 / 28.5967	130.985 / 72.2233

ESTIMATOR: LEAST SQUARES USING LINEAR RESIDUAL FROM BELAR(1) RANDOM COEFFICIENT PROCESS; RHO = 0.31905.

*** WIDEST Y VALUES FOUND: YMIN=-.6165   YMAX= 1.253

TABLE III.E.4.3

SIMTBED Summary Statistics for Estimating γ by the Least Squares Estimator, $\hat{\gamma}_{LS}$, in the BELAR(1) Process with α=.55 and γ=-.67970

SUMMARY STATISTICS (MEAN/STD) 10 REPETITIONS

SUBSAMPLE SIZE	25	50	75	100	125	175	250	500
MEAN	-0.6170 0.3987E-03	-0.6428 0.6671E-03	-0.6512 0.6002E-03	-0.6587 0.9434E-03	-0.6612 0.5731E-03	-0.6658 0.5543E-03	-0.6696 0.5249E-03	-0.6741 0.5349E-03
STD	0.1789 0.3443E-03	0.1279 0.4158E-03	0.1062 0.3566E-03	0.9205E-01 0.4329E-03	0.8274E-01 0.2287E-03	0.7207E-01 0.7372E-03	0.5947E-01 0.4228E-03	0.4315E-01 0.3016E-03
SKEWNESS	0.7375 0.1484E-01	0.6535 0.2479E-01	0.5502 0.1785E-01	0.4987 0.1325E-01	0.4033 0.2621E-01	0.3724 0.2213E-01	0.3170 0.2944E-01	0.1929 0.2149E-01
KURTOSIS	0.8139 0.7653E-01	0.6297 0.8529E-01	0.4556 0.6677E-01	0.4605 0.6978E-01	0.2744 0.7109E-01	0.3261 0.7291E-01	0.3341 0.6369E-01	0.9548E-01 0.1232
SER.COR.	-0.4656E-02 0.3506E-02	-0.7293E-02 0.6261E-02	-0.5211E-02 0.6256E-02	-0.6921E-02 0.6850E-02	-0.1355E-02 0.6224E-02	0.3269E-02 0.8492E-02	-0.2930E-02 0.9226E-02	-0.6288E-02 0.1412E-01
QUANTILES								
0.010	-0.9302 0.9927E-03	-0.8778 0.2179E-02	-0.8547 0.1415E-02	-0.8403 0.1358E-02	-0.8291 0.1286E-02	-0.8130 0.2682E-02	-0.7974 0.1437E-02	-0.7690 0.1456E-02
0.025	-0.8930 0.9091E-03	-0.8515 0.1452E-02	-0.8311 0.1235E-02	-0.8183 0.1231E-02	-0.8085 0.1174E-02	-0.7964 0.1804E-02	-0.7781 0.1029E-02	-0.7545 0.9820E-03
0.050	-0.8617 0.6750E-03	-0.8260 0.1400E-02	-0.8079 0.4937E-03	-0.7970 0.8674E-03	-0.7865 0.1209E-02	-0.7777 0.1265E-02	-0.7628 0.9711E-03	-0.7435 0.1045E-02
0.100	-0.8232 0.6262E-03	-0.7946 0.9546E-03	-0.7795 0.6905E-03	-0.7711 0.9874E-03	-0.7634 0.1009E-02	-0.7553 0.1385E-02	-0.7437 0.6002E-03	-0.7284 0.6510E-03
0.250	-0.7472 0.3712E-03	-0.7347 0.6429E-03	-0.7276 0.7494E-03	-0.7238 0.9601E-03	-0.7186 0.5823E-03	-0.7155 0.1050E-02	-0.7103 0.6346E-03	-0.7045 0.6429E-03
0.500	-0.6418 0.7974E-03	-0.6574 0.8795E-03	-0.6612 0.7279E-03	-0.6663 0.1040E-02	-0.6668 0.6510E-03	-0.6701 0.6429E-03	-0.6727 0.8426E-03	-0.6748 0.7206E-03
0.750	-0.5137 0.7634E-03	-0.5666 0.8915E-03	-0.5861 0.8795E-03	-0.6014 0.1123E-02	-0.6104 0.6671E-03	-0.6212 0.8489E-03	-0.6313 0.8974E-03	-0.6460 0.9033E-03
0.900	-0.3751 0.1218E-02	-0.4740 0.1358E-02	-0.5100 0.1330E-02	-0.5377 0.1537E-02	-0.5524 0.8974E-03	-0.5715 0.7494E-03	-0.5924 0.1261E-02	-0.6184 0.1085E-02
0.950	-0.2842 0.1177E-02	-0.4090 0.1459E-02	-0.4603 0.1804E-02	-0.4957 0.2485E-02	-0.5155 0.1080E-02	-0.5388 0.1430E-02	-0.5676 0.1187E-02	-0.6008 0.8855E-03
0.975	-0.2051 0.2135E-02	-0.3497 0.3179E-02	-0.4172 0.1967E-02	-0.4582 0.3305E-02	-0.4817 0.1598E-02	-0.5123 0.1870E-02	-0.5450 0.2125E-02	-0.5854 0.1558E-02
0.990	-0.1095 0.3683E-02	-0.2848 0.4105E-02	-0.3575 0.2001E-02	-0.4092 0.2708E-02	-0.4418 0.3629E-02	-0.4748 0.2602E-02	-0.5138 0.3541E-02	-0.5664 0.1783E-02

MEAN OF REGRESSION ON AVERAGES - COEFFICIENTS:	-0.679071 0.100859E-02	2.54827 0.285027	-47.8256 19.2260	566.215 324.715
STD DEV OF REGRESSION - COEFFICIENTS:	0.294653E-02 0.194584E-03	0.751958 0.554202E-01	47.3255 3.58258	769.073 59.9415
REGRESSION ON VARIANCE - COEFFICIENTS:	1.00983 0.469854E-01	-2.01436 1.44020	4.46633 12.9963	1.79216 35.0460

ESTIMATOR: LEAST SQUARES USING LINEAR RESIDUAL FROM BELAR(1) RANDOM COEFFICIENT PROCESS; RHO = -0.6797.

*** WIDEST Y VALUES FOUND: YMIN=-1.470 YMAX=0.7451

results reflect the theoretical behavior of the estimator as derived above.

We note that the joint conditional least squares estimators of μ and γ in the BELAR(1) process are the same as in the linear AR(1) processes. Minimizing the sum $\sum_{i=2}^{n} R_i^2$ where now

$$R_i = (X_i-\mu) - \gamma(X_{i-1}-\mu), \tag{III.E.4.12}$$

leads to the following joint estimators for μ and γ

$$\hat{\mu} = \left(\sum_{i=2}^{n} X_i - \hat{\gamma} \sum_{i=2}^{n} X_{i-1} \right) / (n-1)(1-\hat{\gamma}), \tag{III.E.4.13}$$

$$\hat{\gamma} = \sum_{i=2}^{n} (X_i-\hat{\mu})(X_{i-1}-\hat{\mu}) \Big/ \sum_{i=2}^{n} (X_{i-1}-\hat{\mu})^2. \tag{III.E.4.14}$$

For large n these equations reduce to the familiar ones

$$\hat{\mu} = \overline{X} \tag{III.E.4.15}$$

$$\hat{\gamma} = \sum_{i=2}^{n} (X_i-\overline{X})(X_{i-1}-\overline{X}) \Big/ \sum_{i=2}^{n} (X_{i-1}-\overline{X})^2. \tag{III.E.4.16}$$

We now turn in the next section to the question of alternative estimators for γ given that $\mu = 0$ and $\lambda = 1$.

5. Other Estimators of the Lag-1 Serial Correlation

a. Estimators Based on a Non-linear Residual

In this section, we explore other possibilities for estimating γ in the BELAR(1) process. There is a question as to why one should use the linear residual since the BELAR(1) process is a random coefficient process which is non-linear. Secondly, why should you minimize the square of the linear residual as opposed to minimizing some other symmetric loss function which is a function of the linear residual? The answer to both questions is that the least squares estimator of γ based on the linear residual out-performed other estimators in the simulation experiment.

Consider the following types of non-linear residuals

$$R_n^* = X_n - \gamma X_{n-1} - \beta(X_n^2 - 2), \qquad \text{(III.E.5.1)}$$

$$R_n' = X_n - \gamma X_{n-1} - \beta X_{n-1}^2 \operatorname{Sign}(X_{n-1}). \qquad \text{(III.E.5.2)}$$

From (III.E.5.1), it follows that R_n^* has zero mean and

$$\operatorname{Var}(R_n^*) = 2(1-\gamma^2+10\beta^2), \qquad \text{(III.E.5.3)}$$

$$\operatorname{Cov}(R_n^*, R_{n-1}^*) = 20\alpha\beta^2. \qquad \text{(III.E.5.4)}$$

Introducing the extra parameter, β, makes the residuals, R_n^*, correlated unless $\alpha = 0$ or $\beta = 0$. If β is zero, then we again have the usual linearized residual in (III.E.4.4). If $\beta^2 = \gamma^2/10$, then the variance is a constant, but the residuals are still correlated. It is easy to

compute the least squares estimators for γ and β from (III.E.5.1) and (III.E.5.2). We simulated the estimators of γ and β and compared them to the results based on (III.E.4.4) with $\beta = 0$. From Table III.E.5.1, we see that the different estimators of γ from all three residuals are close to the true γ. The result is that the estimates of β are very close to zero.

To see how much the value of γ could change with β fixed at some non-zero values, we simulated the least squares estimator of γ with $\beta = 0$ and the estimator of γ based on (III.E.5.1) with $\beta = \gamma\sqrt{10}$ and again with $\beta = -\gamma/\sqrt{10}$. From Table III.E.5.2, we see that $\beta \neq 0$ severely alters the estimate of the serial correlation. Therefore, in the remainder of this subsection, we consider alternative estimators for γ in the BELAR(1) process to be only those based on the linear residual.

b. Estimators Based on the Linear Residual, R_n

Besides the asymptotically unbiased least squares estimator, we considered the following well-known estimators of γ in linear AR(1) models:

1) The Huber(c) function as described by Denby and Martin [Ref. 38].

The estimator, $\hat{\gamma}_H$, is the value of γ that satisfies the equation

$$\sum_{i=2}^{n} x_{i-1}\Psi_H(x_i-\gamma x_{i-1}) = 0, \qquad \text{(III.E.5.5)}$$

TABLE III.3.5.1

Simulation Results for Various Definitions of R_n in BELAR(1)

1. N = 500 $\alpha = .5$ $\gamma = \mathrm{Corr}(X_n, X_{n-1}) \approx .63662$

DATA	$\hat{\gamma}_{LS}$	$\beta = 0$	$\hat{\gamma}^*$	$\hat{\beta}^*$	$\hat{\gamma}'$	$\hat{\beta}'$
X1	.56891	0	.57192	.00279	.62082	-.01771
X2	.61996	0	.61630	-.00815	.56054	+.01637
X3	.62651	0	.62604	.00358	.78189	-.05808
X4	.57995	0	.58374	-.01865	.75716	-.07208
X5	.59236	0	.59233	-.02100	.70995	-.04748
AVG	.59754		.59807	-.00829	.68607	-.03580
STD	.02499		.02257	.01154	.09330	.03535
BIAS	-.03908		-.03855	-.00829	+.04945	-.03580

2. N = 1000 $\alpha = .5$ $\gamma = \mathrm{Corr}(X_n, X_{n-1}) \approx .63662$

DATA	$\hat{\gamma}_{LS}$	$\beta = 0$	$\hat{\gamma}^*$	$\hat{\beta}^*$	$\hat{\gamma}'$	$\hat{\beta}'$
Y1	.63026	0	.62955	-.00423	.62985	.00013
Y2	.67422	0	.65653	.02520	.59178	.03095
Y3	.62566	0	.62921	-.00590	.59646	.01093
Y4	.67738	0	.67777	.00233	.60522	.02359
Y5	.64664	0	.64784	-.00560	.62841	.00581
AVG	.65083		.64818	.00236	.61034	.01428
STD	.02411		.02032	.01320	.01782	.01273
BIAS	+.01421		+.01156	+.00236	-.02628	+.01428

3. N = 1500 $\alpha = .75$ $\gamma = \mathrm{Corr}(X_n, X_{n-1}) \approx .83463$

DATA	$\hat{\gamma}_{LS}$	$\beta = 0$	$\hat{\gamma}^*$	$\hat{\beta}^*$	$\hat{\gamma}'$	$\hat{\beta}'$
Z1	.81183	0	.81671	.00797	.86364	-.01821
Z2	.80699	0	.80700	-.00040	.82072	-.00511
Z3	.81777	0	.81795	-.00160	.83399	-.00641
Z4	.85279	0	.85569	-.00728	.89116	-.00193
AVG	.82235		.82434	-.00033	.85238	-.01041
STD	.02077		.02147	.00629	.03147	.00598
BIAS	-.01229		-.01029	-.00033	+.01775	-.01041

TABLE III.E.5.2
Simulation Results for Various Definitions of R_n to Estimate γ Given β in BELAR(1)

N = 500; $\alpha = .5$ $\gamma = .63662$

DATA	$(\hat{\gamma}_{LS} \mid \beta = 0)$	$\left(\hat{\gamma}^* \mid \beta = \frac{\gamma}{\sqrt{10}}\right)$	$\left(\hat{\gamma}^* \mid \beta = \frac{-\gamma}{\sqrt{10}}\right)$
1	.56891	.27552	.27410
2	.61996	.21515	.26257
3	.62695	.38621	.38450
4	.57995	.34356	.39730
5	.59236	.36082	.40557

where

$$\Psi_H(t) = \begin{cases} t & \text{if } |t| \leq c, \\ c \operatorname{Sign}(t) & \text{if } |t| > c. \end{cases} \tag{III.E.5.6}$$

The corresponding weight function $w_H(t)$ is $\Psi_H(t)/t$ and c is a tuning constant. As c goes to infinity $\Psi_H(t)$ approaches t and $\hat{\gamma}_H$ is the least squares estimator of γ. If c = 0, we have the solution of (III.E.5.5) is the median of X_i/X_{i-1}.

For c other than 0 or ∞, there is no closed-form solution to (III.E.5.5). We obtain the Huber(c) estimator of γ by iterating the following scheme:

$$\hat{\gamma}_{k+1} = \frac{\sum_{i=2}^{n} x_i x_{i-1} \, w_H\left(\frac{x_i - \hat{\gamma}_k x_{i-1}}{S_r}\right)}{\sum_{i=2}^{n} x_{i-1}^2 \, w_H\left(\frac{x_i - \hat{\gamma}_k x_{i-1}}{S_r}\right)}, \tag{III.E.5.7}$$

where $\hat{\gamma}_1$ is the least squares estimator of γ and

$$S_r = \frac{\text{median } |X_i|}{.69315}, \qquad \text{(III.E.5.8)}$$

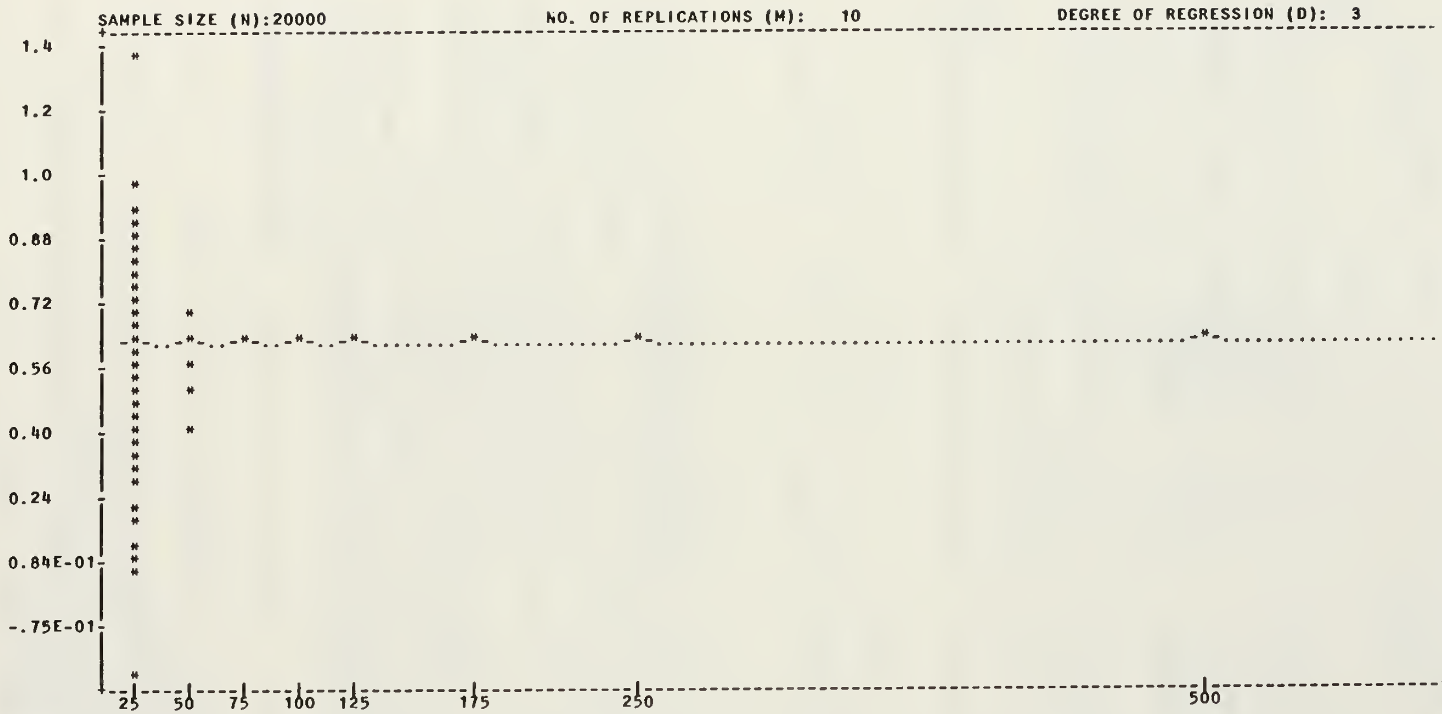
is the scaling constant for the R_i. If $\gamma = 0$, then S_r is the median absolute deviation estimator of the scale parameter in the Laplace distribution as given in Section III.E.3. Typical values of c are 1, 1.5, 2. We use for illustration c = 1 in the simulation along with $\hat{\gamma}_{LS}$, the least squares estimate, and $\hat{\gamma}_M$, the median (X_i/X_{i-1}).

2) The Least Absolute Deviation (LAD) estimator of γ is the minimizer of

$$\sum_{i=2}^{n} |x_i - \gamma x_{i-1}|. \qquad \text{(III.E.5.9)}$$

The solution is, $\hat{\gamma}_{WM}$, the weighted median of x_i/x_{i-1} where the weights are x_{i-1} for i = 2,...,n.

Denby and Martin [Ref. 38] reported that the Huber(c) estimates are consistent and asymptotically unbiased for linear AR(1) models. Bloomfield and Steiger [Ref. 39] showed that the LAD estimator is strongly consistent and asymptotically unbiased for linear AR(1) models. In Figures III.E.5.1 - III.E.5.4 are examples from SIMTBED of the behavior of these estimators in simulated data from LAR(1), a linear AR(1) model with Laplacian marginals and AR(1) correlation structure given in Chapter II. These results appear to be consistent with the results reported above for linear AR(1) processes. The leading coefficient in the expansion for the mean of each estimator does not

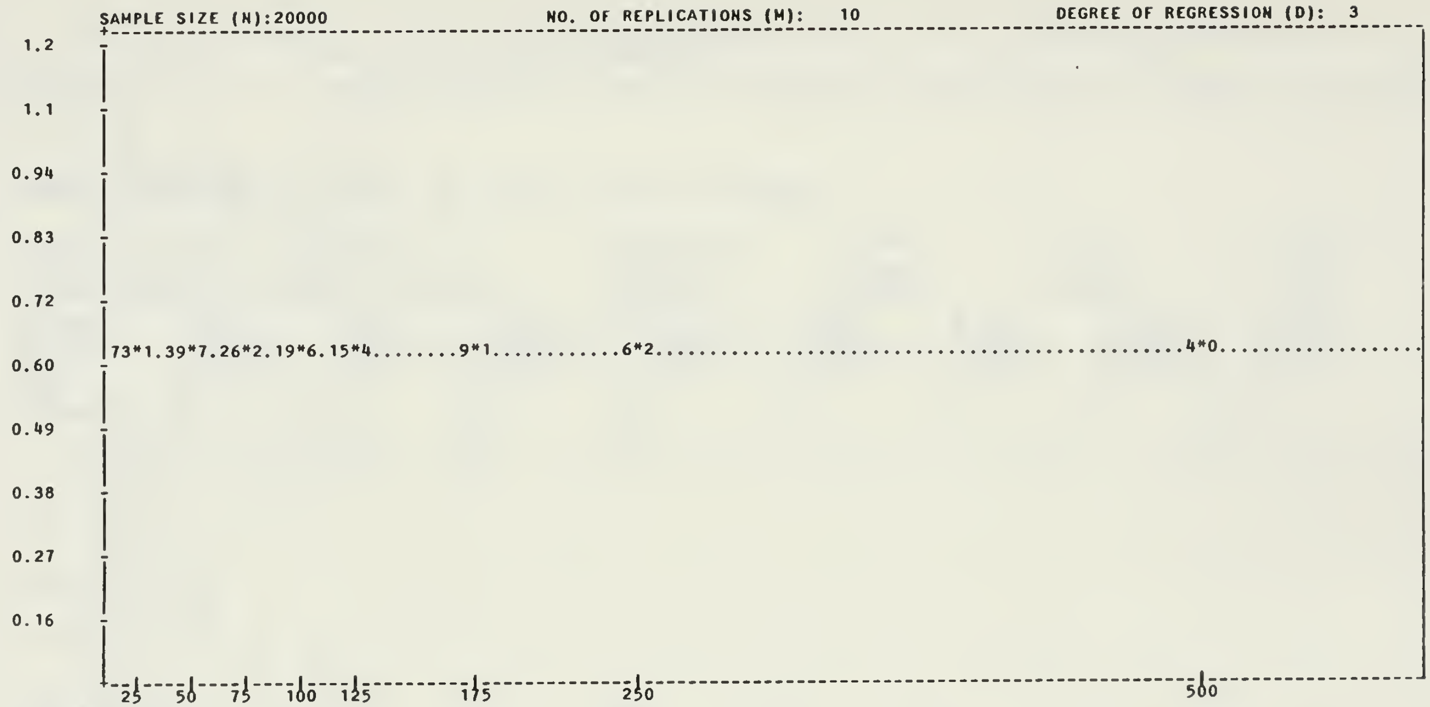


SUBSAMPLE SIZE	25	50	75	100	125	175	250	500
MEAN	0.6344	0.6364	0.6365	0.6365	0.6365	0.6365	0.6365	0.6366
STD	0.3228E-01	0.4552E-02	0.1242E-03	0.1177E-03	0.1104E-03	0.9642E-04	0.7608E-04	0.7498E-05
STD MEAN	0.3609E-03	0.7198E-04	0.2408E-05	0.2631E-05	0.2759E-05	0.2856E-05	0.2690E-05	0.3749E-06
SKEWNESS	-6.1529	-31.8662	1.0006	1.0007	1.0009	1.0013	1.0019	1.0038
KURTOSIS	180.9922	1451.0112	-2.0015	-2.0020	-2.0025	-2.0035	-2.0050	-2.0100

MEAN OF REGRESSION ON AVERAGES - COEFFICIENTS:	0.636617	-0.263561E-02	0.609372	-36.8116
VARIANCE OF REGRESSION - COEFFICIENTS:	0.529819E-07	0.105641E-03	0.884593	288.122
STD DEV OF REGRESSION - COEFFICIENTS:	0.230178E-03	0.102782E-01	0.940528	16.9742
REGRESSION ON VARIANCE - COEFFICIENTS:	-0.153156E-01	0.566167	-6.50287	23.5301

ESTIMATOR: MEDIAN X(I)/X(I-1) FROM LAR(1) PROCESS. RHO = 0.63662.
VERTICAL SCALE: YMIN = -0.2027
YMAX = 1.3558

Figure III.E.5.1. SIMTBED Boxplot Analysis of Median (X_i/X_{i-1}) Estimator of γ with $\gamma=.63662$ in the LAR(1) Process

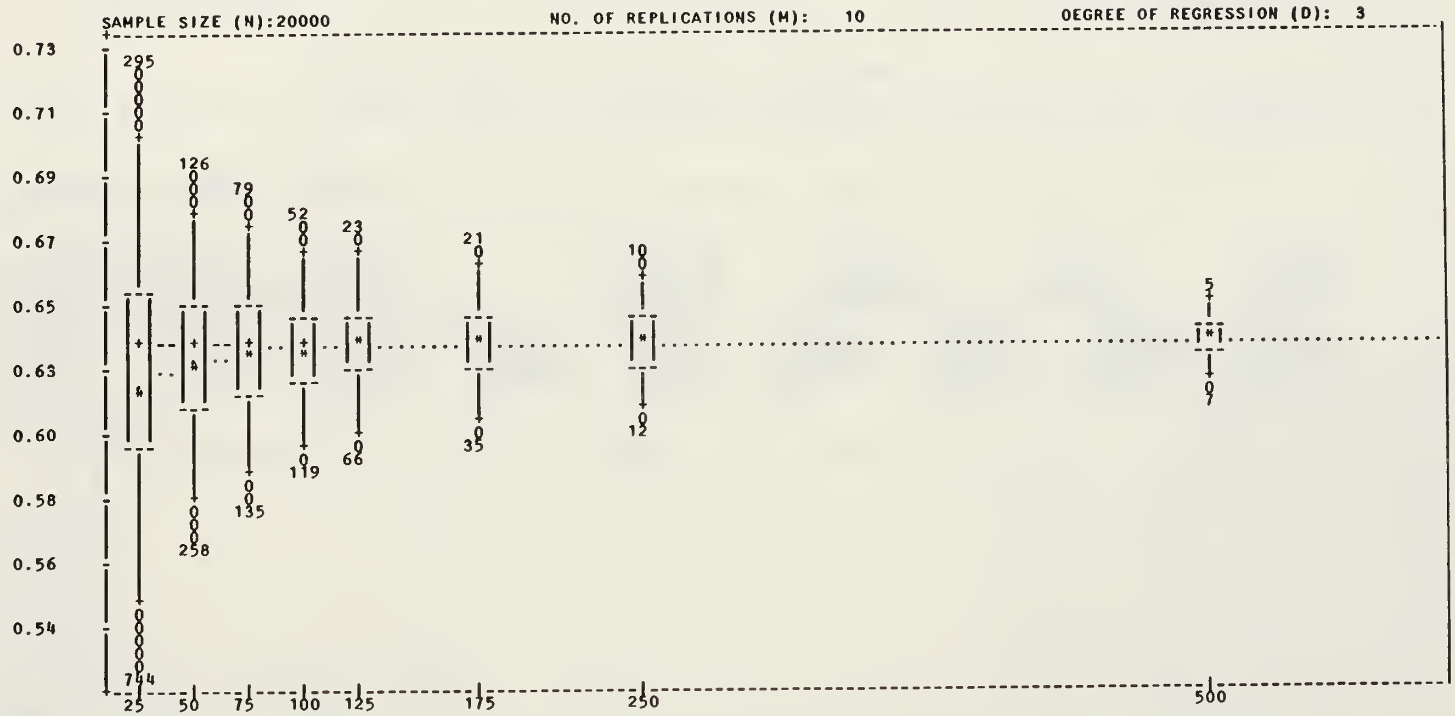


SUBSAMPLE SIZE	25	50	75	100	125	175	250	500
MEAN	0.6271	0.6354	0.6362	0.6364	0.6365	0.6365	0.6365	0.6366
STD	0.5813E-01	0.1719E-01	0.6603E-02	0.3771E-02	0.6872E-03	0.2936E-03	0.7593E-04	0.7488E-05
STD MEAN	0.6500E-03	0.2718E-03	0.1280E-03	0.8432E-04	0.1718E-04	0.8696E-05	0.2685E-05	0.3744E-06
SKEWNESS	-5.2489	-14.2724	-17.1407	-23.2150	-37.9803	-25.8877	1.0019	1.0042
KURTOSIS	43.6662	279.9202	492.2214	651.2222	1490.5356	790.8833	-2.0050	-2.0088

MEAN OF REGRESSION ON AVERAGES - COEFFICIENTS:	0.636572	0.196543E-01	-1.40293	-114.304
VARIANCE OF REGRESSION - COEFFICIENTS:	0.529819E-07	0.207290E-02	15.7610	5474.89
STD DEV OF REGRESSION - COEFFICIENTS:	0.230178E-03	0.455291E-01	3.97001	73.9925
REGRESSION ON VARIANCE - COEFFICIENTS:	-0.774512E-02	0.404094	-6.62432	34.5494

ESTIMATOR: WEIGHTED MEDIAN X(I)/X(I-1) FROM LAR(1) LINEAR PROCESS; RHO = 0.63362.
VERTICAL SCALE: YMIN = 0.0667
YMAX = 1.1641

Figure III.E.5.2. SIMTBED Boxplot Analysis of Weighted Median (X_i/X_{i-1}) Estimator of γ with γ=.63662 in the LAR(1) Process

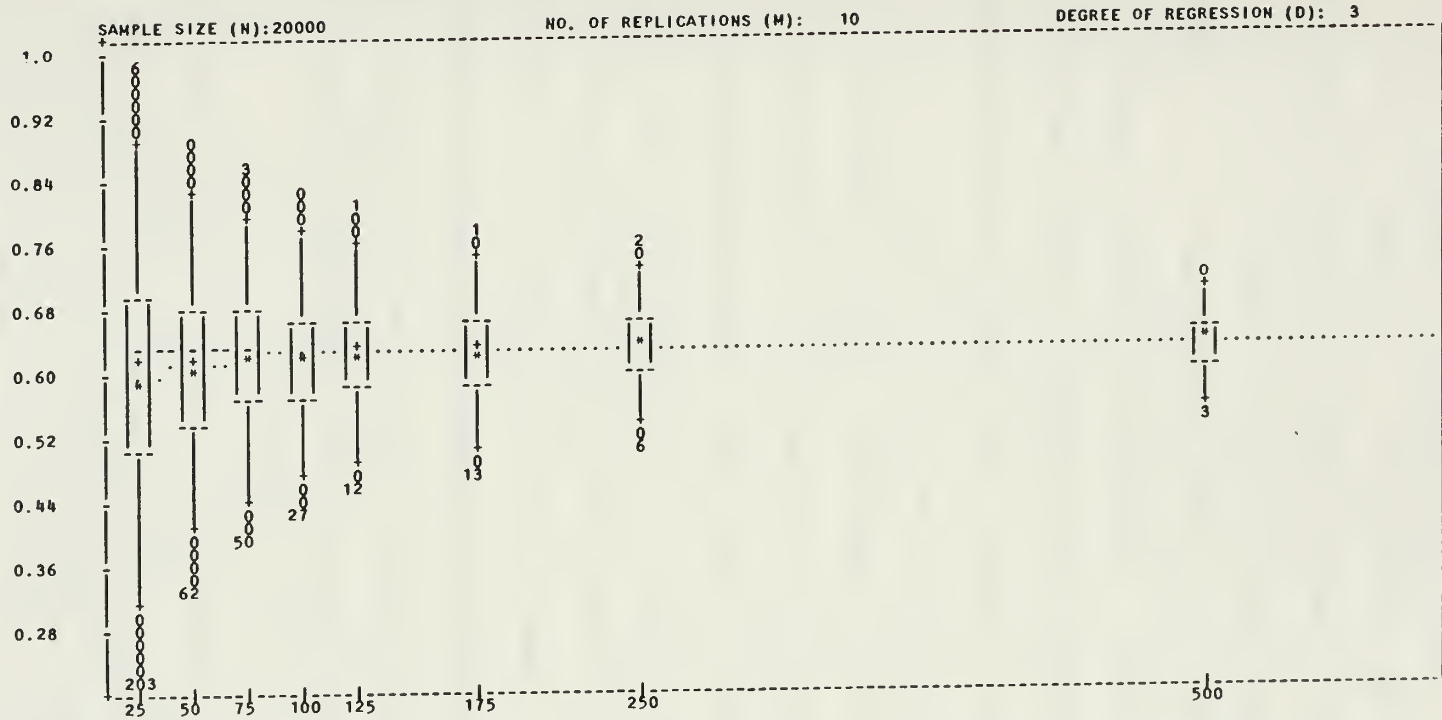


SUBSAMPLE SIZE	25	50	75	100	125	175	250	500
MEAN	0.6180	0.6296	0.6323	0.6332	0.6349	0.6353	0.6356	0.6357
STO	0.8507E-01	0.4267E-01	0.3266E-01	0.2595E-01	0.1931E-01	0.1628E-01	0.1266E-01	0.8475E-02
STO MEAN	0.9511E-03	0.6747E-03	0.6332E-03	0.5802E-03	0.4828E-03	0.4822E-03	0.4477E-03	0.4237E-03
SKEWNESS	-2.0576	-1.6337	-1.4519	-1.5711	-0.7647	-0.6387	-0.3254	-0.5238
KURTOSIS	9.2196	8.3189	8.3799	8.9713	2.9510	3.3905	1.7610	1.9636

MEAN OF REGRESSION ON AVERAGES - COEFFICIENTS:	0.636270	-0.129989	-12.6605	117.900
VARIANCE OF REGRESSION - COEFFICIENTS:	0.667572E-06	0.594828E-01	306.423	87603.0
STD DEV OF REGRESSION - COEFFICIENTS:	0.817050E-03	0.243891	17.5049	295.978
REGRESSION ON VARIANCE - COEFFICIENTS:	-0.172461E-01	1.37326	-9.51650	38.0021

ESTIMATOR: ROBUST HUBER(C) C = 1.0. RHO = 0.63662. LAR(1) LINEAR PROCESS.
VERTICAL SCALE: YMIN = 0.5226
YMAX = 0.7329

Figure III.E.5.3. SIMTBED Boxplot Analysis of Huber(c) Estimator of γ with γ=.63662 and c=1 in the LAR(1) Process



SUBSAMPLE SIZE	25	50	75	100	125	175	250	500
MEAN	0.5939	0.6143	0.6207	0.6245	0.6300	0.6302	0.6316	0.6323
STD	0.1586	0.1082	0.9190E-01	0.7749E-01	0.6681E-01	0.5716E-01	0.4942E-01	0.3229E-01
STD MEAN	0.1774E-02	0.1711E-02	0.1782E-02	0.1733E-02	0.1670E-02	0.1693E-02	0.1747E-02	0.1649E-02
SKEWNESS	-0.8646	-0.6954	-0.6712	-0.5962	-0.4308	-0.3508	-0.3564	-0.2780
KURTOSIS	1.1683	0.5747	0.8910	1.2671	0.2580	0.2837	0.3226	-0.1191

MEAN OF REGRESSION ON AVERAGES - COEFFICIENTS:	0.634121	-0.508326	-38.6555	661.164
VARIANCE OF REGRESSION - COEFFICIENTS:	0.672870E-05	0.536488	2750.31	800827.
STD DEV OF REGRESSION - COEFFICIENTS:	0.259397E-02	0.732454	52.4434	894.890
REGRESSION ON VARIANCE - COEFFICIENTS:	0.390075	5.05937	-40.1239	103.961

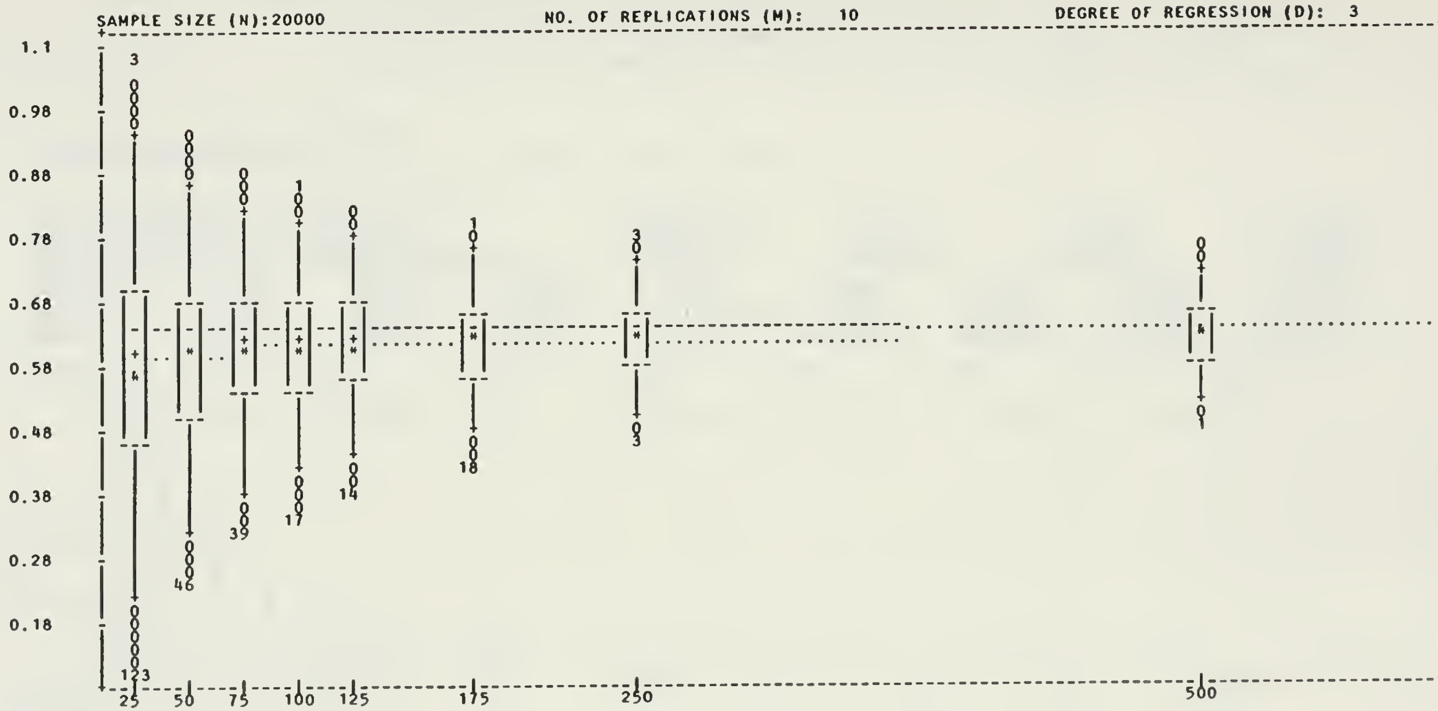
ESTIMATOR: LEAST SQUARES USING LINEAR RESIDUAL FROM LAR(1) LINEAR PROCESS. RHO = 0.63662.
VERTICAL SCALE: YMIN = 0.2146
YMAX = 0.9985

Figure III.E.5.4. SIMTBED Boxplot Analysis of the Least Squares Estimator of γ with γ=.63662 in the LAR(1) Process

differ significantly from the true value, 0.63662. We also see that the median (X_i/X_{i-1}) and the weighted median (X_i/X_{i-1}) estimators are considerably more efficient than either the Huber(c) estimator in Figure III.E.5.3 or the least squares estimator ($c = \infty$) in Figure III.E.5.4.

Since the least squares estimator remains asymptotically unbiased for the BELAR(1) process as was shown in Section III.E.4, it was of interest to observe how the Huber(c) estimators, $c < \infty$, and the LAD estimator of γ would behave. Considering the ordering suggested by the simulation results in the LAR(1) process, it would seem possible that the Huber(c) estimates could be better than the least squares estimator of γ. In the boxplot analyses in Figures III.E.5.5 - III.E.5.8 are the results of the simulation for $\gamma = .63662$, but for data from the BELAR(1) process. The boxplots in Figure III.E.5.5 display the theoretical behavior of the least squares estimator of γ. The other estimators of γ appear to be converging to other values $\gamma_0 \neq \gamma$. To see this, note the first entry in the coefficients for the asymptotic expansion of the mean of $\hat{\gamma}$ in Figures III.E.5.6 - III.E.5.8. In each case $\gamma_0 > \gamma$. Also from the estimate of the standard deviation, we assert that γ_0 is significantly larger than γ for each of the alternative estimators investigated here, because the difference, $|\gamma - \gamma_0|$, is larger than four standard deviations.

For the BELAR(1) process, we observe a reversal from the LAR(1) process in preference for the estimator of γ. We will use the least squares estimator as the initial estimator of γ in the iterative procedure for finding the maximum likelihood estimator of γ which we develop next.

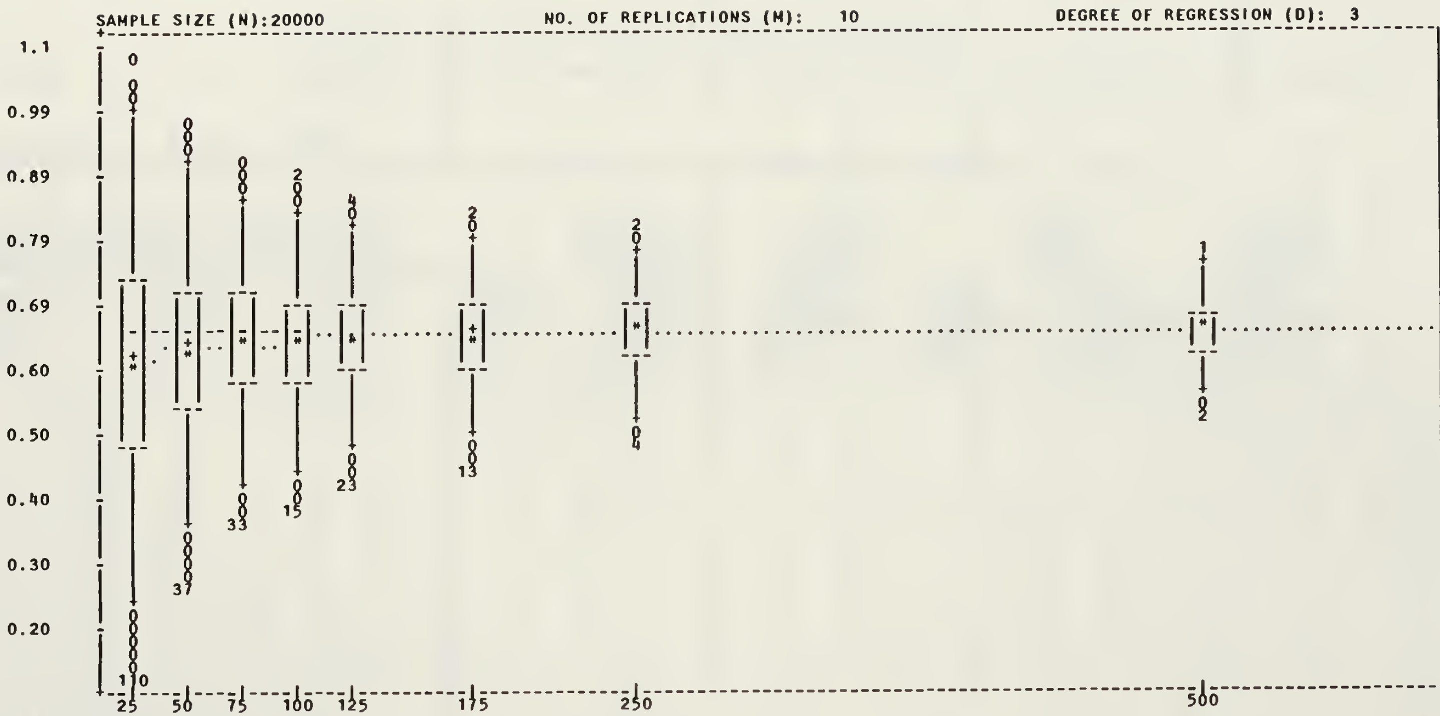


SUBSAMPLE SIZE	25	50	75	100	125	175	250	500
MEAN	0.5758	0.6000	0.6118	0.6144	0.6189	0.6222	0.6275	0.6310
STD	0.1835	0.1346	0.1111	0.9558E-01	0.8674E-01	0.7577E-01	0.6231E-01	0.4859E-01
STD MEAN	0.2052E-02	0.2128E-02	0.2154E-02	0.2137E-02	0.2169E-02	0.2244E-02	0.2203E-02	0.2430E-02
SKEWNESS	-0.7441	-0.5269	-0.5686	-0.4242	-0.4322	-0.3876	-0.1637	-0.1529
KURTOSIS	0.8197	0.3870	0.3500	0.2574	0.2841	0.6862	0.1140	-0.2645

MEAN OF REGRESSION ON AVERAGES - COEFFICIENTS:	0.635392	-2.31403	36.8632	-402.901
VARIANCE OF REGRESSION - COEFFICIENTS:	0.124402E-04	0.918215	3838.85	0.102546E+07
STD DEV OF REGRESSION - COEFFICIENTS:	0.352706E-02	0.958236	61.9585	1012.65
REGRESSION ON VARIANCE - COEFFICIENTS:	1.67258	-16.7334	122.009	-295.576

ESTIMATOR: LEAST SQUARES USING LINEAR RESIDUAL FROM BETA-LAPLACE AR(1) RANDOM COEFF PROCESS; RHO = 0.63662.
VERTICAL SCALE: YMIN = 0.1000
YMAX = 1.0787

Figure III.E.5.5. SIMTBED Boxplot Analysis of the Least Squares Estimator of γ with α=.5 and γ=.63662 in the BELAR(1) Process

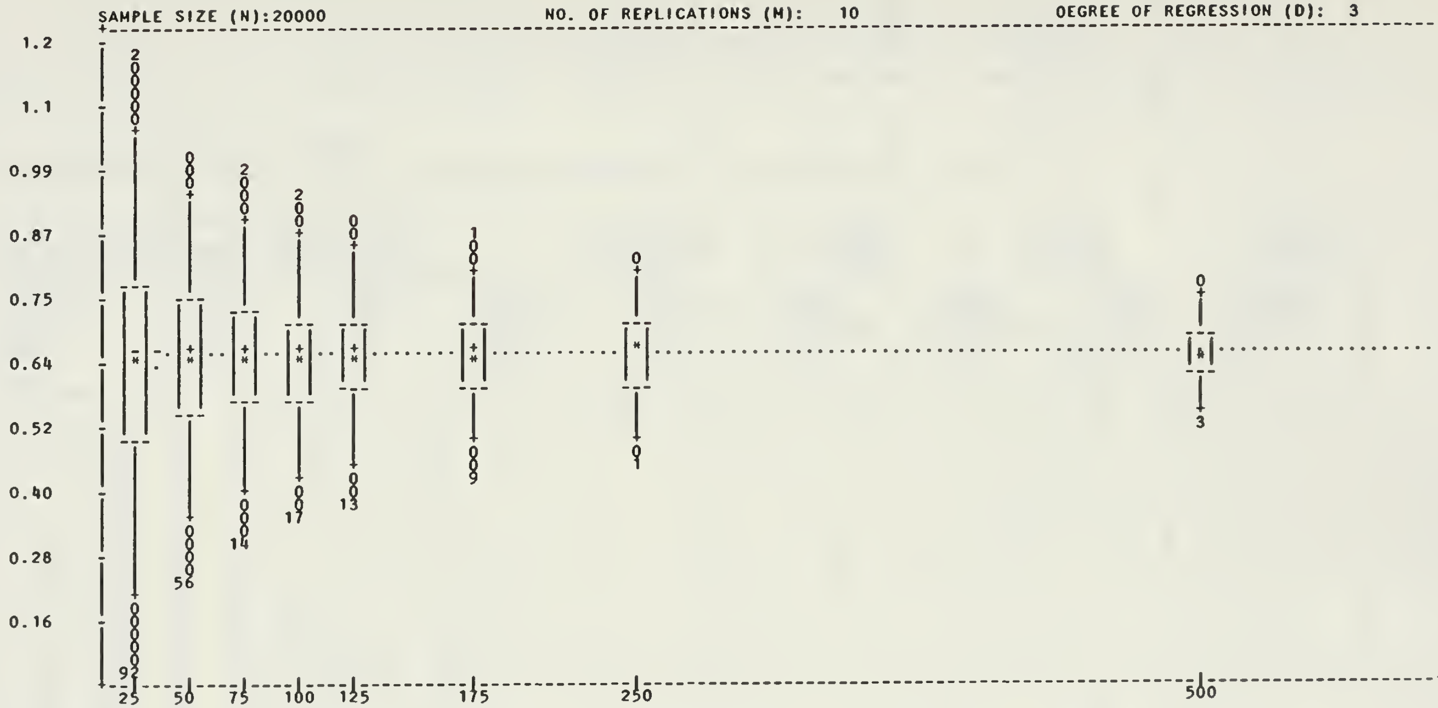


SUBSAMPLE SIZE	25	50	75	100	125	175	250	500
MEAN	0.6063	0.6314	0.6405	0.6420	0.6466	0.6489	0.6543	0.6557
STD	0.1860	0.1336	0.1092	0.9499E-01	0.8624E-01	0.7421E-01	0.6160E-01	0.4732E-01
STD MEAN	0.2079E-02	0.2112E-02	0.2118E-02	0.2124E-02	0.2156E-02	0.2198E-02	0.2178E-02	0.2366E-02
SKEWNESS	-0.6383	-0.4794	-0.5115	-0.3478	-0.4358	-0.3575	-0.1621	-0.2141
KURTOSIS	0.3508	0.1253	0.2574	0.0682	0.5928	0.5524	0.0225	-0.0599

MEAN OF REGRESSION ON AVERAGES - COEFFICIENTS:	0.660213	-2.04271	44.6986	-678.037
VARIANCE OF REGRESSION - COEFFICIENTS:	0.121540E-04	1.08527	4713.99	0.128226E+07
STD DEV OF REGRESSION - COEFFICIENTS:	0.348627E-02	1.04176	68.6585	1132.37
REGRESSION ON VARIANCE - COEFFICIENTS:	1.51909	-13.4174	94.7185	-219.952

ESTIMATOR: ROBUST HUBER(C) C = 1.0. RHO = 0.63662. BELAR(1) PROCESS.
VERTICAL SCALE: YMIN = 0.1254
YMAX = 1.0847

Figure III.E.5.6. SIMTBED Boxplot Analysis of the Huber(c) Estimator of γ with α=.5, γ=.63662 and c=1 in the BELAR(1) Process

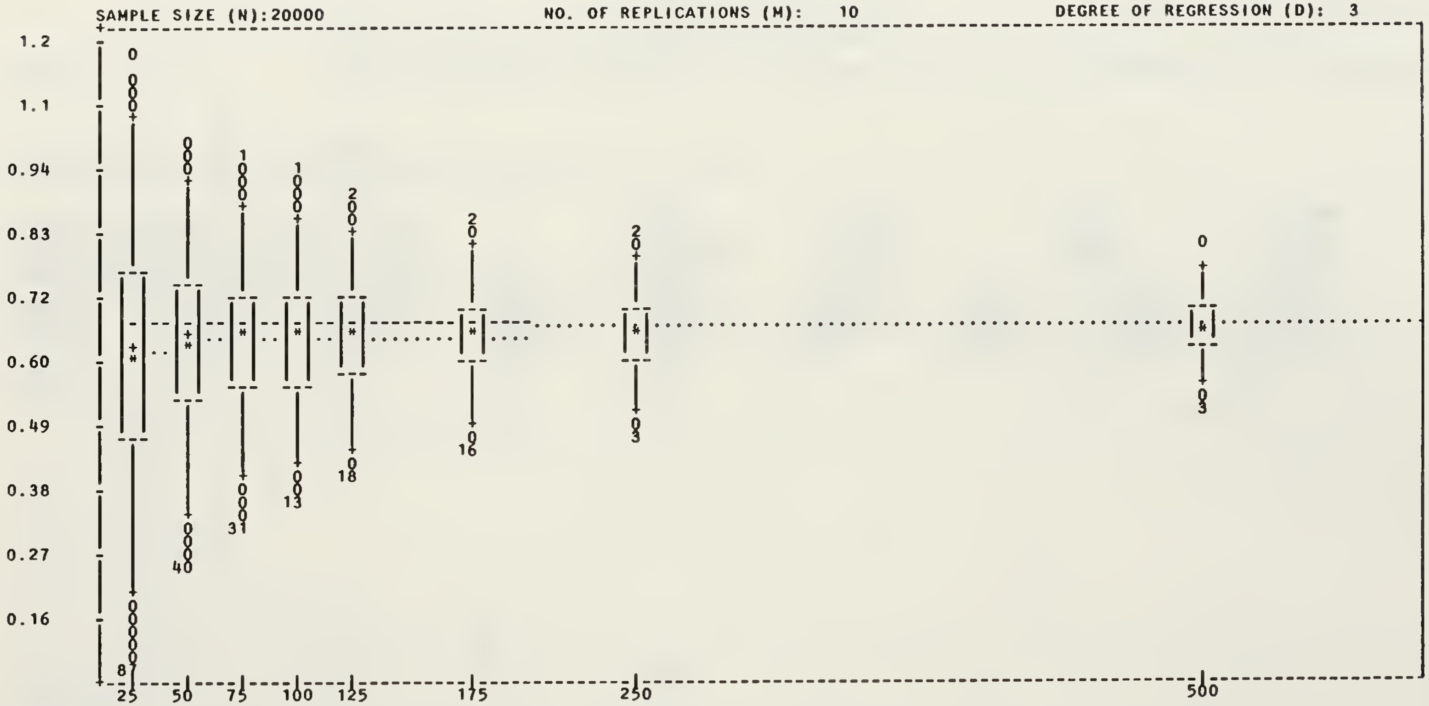


SUBSAMPLE SIZE	25	50	75	100	125	175	250	500
MEAN	0.6365	0.6507	0.6553	0.6547	0.6571	0.6567	0.6612	0.6583
STD	0.2158	0.1534	0.1234	0.1119	0.9779E-01	0.8142E-01	0.6917E-01	0.5004E-01
STO MEAN	0.2412E-02	0.2425E-02	0.2392E-02	0.2502E-02	0.2445E-02	0.2412E-02	0.2445E-02	0.2502E-02
SKEWNESS	-0.5780	-0.5432	-0.3541	-0.3664	-0.3428	-0.4440	-0.0785	-0.0578
KURTOSIS	0.4361	0.2620	-0.0159	0.1916	0.0684	0.5314	-0.2426	-0.0041

MEAN OF REGRESSION ON AVERAGES - COEFFICIENTS:	0.660092	-0.329107	-7.00634	18.7444
VARIANCE OF REGRESSION - COEFFICIENTS:	0.559489E-05	0.622558	2794.82	741196.
STO OEV OF REGRESSION - COEFFICIENTS:	0.236535E-02	0.789024	52.8661	860.927
REGRESSION ON VARIANCE - COEFFICIENTS:	1.26003	-0.930313	1.92180	1.64526

ESTIMATOR: MEDIAN X(I)/X(I-1) FROM BELAR(1) PROCESS; RHO = 0.63662.
VERTICAL SCALE: YMIN = 0.0671
YMAX = 1.2289

Figure III.E.5.7. SIMTBED Boxplot Analysis of Median (X_i/X_{i-1}) Estimator of γ with α=.5 and γ=.63662 in the BELAR(1) Process



SUBSAMPLE SIZE	25	50	75	100	125	175	250	500
MEAN	0.6130	0.6385	0.6491	0.6501	0.6557	0.6574	0.6643	0.6645
STD	0.2057	0.1462	0.1200	0.1058	0.9459E-01	0.8066E-01	0.6781E-01	0.5254E-01
STD MEAN	0.2300E-02	0.2312E-02	0.2328E-02	0.2367E-02	0.2365E-02	0.2389E-02	0.2397E-02	0.2627E-02
SKEWNESS	-0.5808	-0.5056	-0.4697	-0.3339	-0.4163	-0.3436	-0.1772	-0.2452
KURTOSIS	0.2157	0.1970	0.2151	0.1136	0.3676	0.3657	-0.1493	-0.1165
MEAN OF REGRESSION ON AVERAGES - COEFFICIENTS:				0.669241	-1.96845		31.1528	-421.315
VARIANCE OF REGRESSION - COEFFICIENTS:				0.148137E-04	1.35265		5874.72	0.159679E+07
STD DEV OF REGRESSION - COEFFICIENTS:				0.384886E-02	1.16303		76.6467	1263.64
REGRESSION ON VARIANCE - COEFFICIENTS:				1.80066	-14.8627		100.423	-223.428

ESTIMATOR: WEIGHTED MEDIAN X(I)/X(I-1) FROM BELAR(1) RANDOM COEFFICIENT PROCESS; RHO = 0.63662.
VERTICAL SCALE: YMIN = 0.0667
YMAX = 1.1641

Figure III.E.5.8. SIMTBED Boxplot Analysis of the Weighted Median (X_i/X_{i-1}) Estimator of γ with α=.5 and γ=.63662 in the BELAR(1) Process

6. Maximum Likelihood Estimation of γ

a. Introduction

In this section, we develop the maximum likelihood estimator of the lag-1 serial correlation in the BELAR(1) process, $\hat{\gamma}_{MLE}$. We use the expression for the logarithm of the likelihood function, $L(\alpha)$, in (III.D.2.12) in an iterative procedure to find the values of α and the sign of $A_n^{1/2}(\alpha,1-\alpha)$, that minimizes $-L(\alpha)$; call the pair $(\hat{\alpha}_{MLE}$, sign). Since knowing α and the sign of $A_n^{1/2}(\alpha,1-\alpha)$ uniquely defines γ, $\hat{\gamma}_{MLE}$ can be found from (III.E.4.8) using $(\hat{\alpha}_{MLE}$, sign).

We consider only the univariate problem. That is, we have assumed that $\{X_n\}$ is marginally Laplace distributed or have determined from Q-Q plots that the best ℓ-Laplace fit to the data is when $\ell = 1$. Secondly, we assumed that $\{X_n\}$ is standard Laplace ($\mu = 0$; $\lambda = 1$) or that $\{X_n\}$ has been standardized using a pair of estimators $(\hat{\mu}, \hat{\lambda})$ from Sections III.E.2. and III.E.3.

As a function of α, (III.D.2.12) is very complicated. There is little hope of being able to analytically solve for the critical values of α. In fact, the evaluation of a derivative of (III.D.2.12) is at least as expensive computationally as the function values themselves, since (III.D.2.12) contains exponential functions of α. However, since this is a one-dimensional optimization problem, there are IMSL routines that will perform the search without using deviatives--Golden Section search; bisection method; or interpolation routines.

We chose the IMSL routine ZXLSF which performs a one-dimensional search for a minimum of a smooth function in a closed interval using quadratic interpolation. The FORTRAN routine which

evaluates (III.D.2.12) is formulated so that ZXLSF is searching on the interval (-1,1) where $\alpha < 0$ implies that conditional densities of the form (III.D.2.10) are being evaluated instead of those given by (III.D.2.9) when $\alpha > 0$. The initial value for α to start the iteration procedure of ZXLSF is a four-digit approximation ($\hat{\alpha}_{LS}$, $\hat{\text{sign}}_{LS}$) corresponding to the least squares estimate of serial correlation, $\hat{\gamma}_{LS}$, obtained from (III.E.4.8).

The queston of accuracy in the calculation of (III.D.2.12) is especially important because the likelihood surface is extremely flat in many cases. We want some assurance that ZXLSF is efficiently searching for the optimum and not "chasing roundoff errors". This happened before we increased the accuracy parameter in DCADRE and used double precision. In order to assess the accuracy of our calculations, we constructed first- and second-divided differences for values of α and (III.D.2.12). The divided differences are approximations for the derivatives. For those simulations that we checked, there was one transition of the slope through zero at the critical point found by ZXSLF. The second-divided differences at all points in the vicinity of the critical value were positive indicating the general convex upward shape of (III.D.2.12). Sometimes there was some fluctuation in values of the second-divided differences, but no change of signs near the reported optimum.

The fluctuating values of the second-divided difference indicated some noise in the calculations. This occurred in two places. If the second-divided difference covered points on both sides of $\alpha = 1/2$, then there was often a jump in the value of the second-divided

difference. This occurred because of the change in the method of calculating the conditional density when α changed from $\alpha < .5$ to $\alpha \geq .5$. Other times, slight aberrations in the observed pattern of the second-divided differences occurred for values of α that were small, $0 < \alpha < .15$. This is attributed to the fact that DCADRE evaluations for the table of values of the $(1-\alpha)$-Laplace density $(0 < \alpha < .15)$ in many subintervals was not behaving regularly. The computed value was accepted because the estimated error was small, relative to the accuracy requirements. The important consideration, however, was that no error in calculating (III.D.2.12) should be so large as to falsely indicate a change in convexity in the vicinity of an extremum, so that ZXLSF would be ineffective at locating it.

The selection of a good starting point in this procedure is also important. It is desirable to commence the iteration in ZXLSF as close to the global optimum as possible in order to reduce the possibility of converging to a local optimium. Note, also, that as a function of α, the conditional density is not necessarily convex and often is not even unimodal across the range from $\gamma = +1$ to $\gamma = -1$.

Since (III.D.2.12) is the logarithm of the product of such functions, there is no assurance that (III.D.2.12) has a single relative maximum especially for small sample sizes. When the sample size is small, it is advisable to pick a starting value for the iteration on both sides of $\alpha = 0$. Select the maximum likelihood estimator to be the one with the higher value of $L(\alpha)$ if the routine produces two different $\hat{\alpha}$'s, corresponding to the pairs $(\hat{\alpha}_1,+)$ and $(\hat{\alpha}_2,-)$.

Since we know that $\hat{\gamma}_{LS}$ is a consistent, asymptotically unbiased and asymptotically Normally distributed estimator for γ, we chose the value of α and model corresponding to $\hat{\gamma}_{LS}$ as our initial guess in ZXLSF.

b. Simulation Results

The maximum likelihood routine for estimating γ was tested in simulations using computer generated data from the BELAR(1) process with known parameter values of ℓ, μ, λ and α. By performing M independent simulations of sample size N (where N is increased for each set of M simulations) and fixed α, we were able to compare the standard deviation and bias (if any), of $\hat{\gamma}_{MLE}$ to that of the initial least squares estimator $\hat{\gamma}_{LS}$, for which the asymptotic distribution is Normal. Changes in the Normal plots for one set of M simulations for N small to a second set of M simulations for a larger N would give some indication of how fast $\hat{\gamma}_{MLE}$ is or is not converging to a Normal distribution.

Both M and N were small in the simulations for two reasons. Since the asymptotic distribution of $\hat{\gamma}_{LS}$ was known, it was of more interest to see how much better $\hat{\gamma}_{MLE}$ was for the smaller samples (i.e., was the bias smaller for $\hat{\gamma}_{MLE}$ or was it, in fact, unbiased). Secondly, the run times for calculating (III.D.2.12) for $N < 200$ was long. The evaluation per sample of size $N = 25$ ranged from 100-300 secs. For $N = 175$, the run times ranged from 700-950 secs.

Figures III.E.6.1, III.E.6.2 and III.E.6.3 are the Normal plots of twenty realizations of the maximum likelihood estimator of serial correlation and the least squares estimator of serial correlation for simulated data from the BELAR(1) process for selected values of α

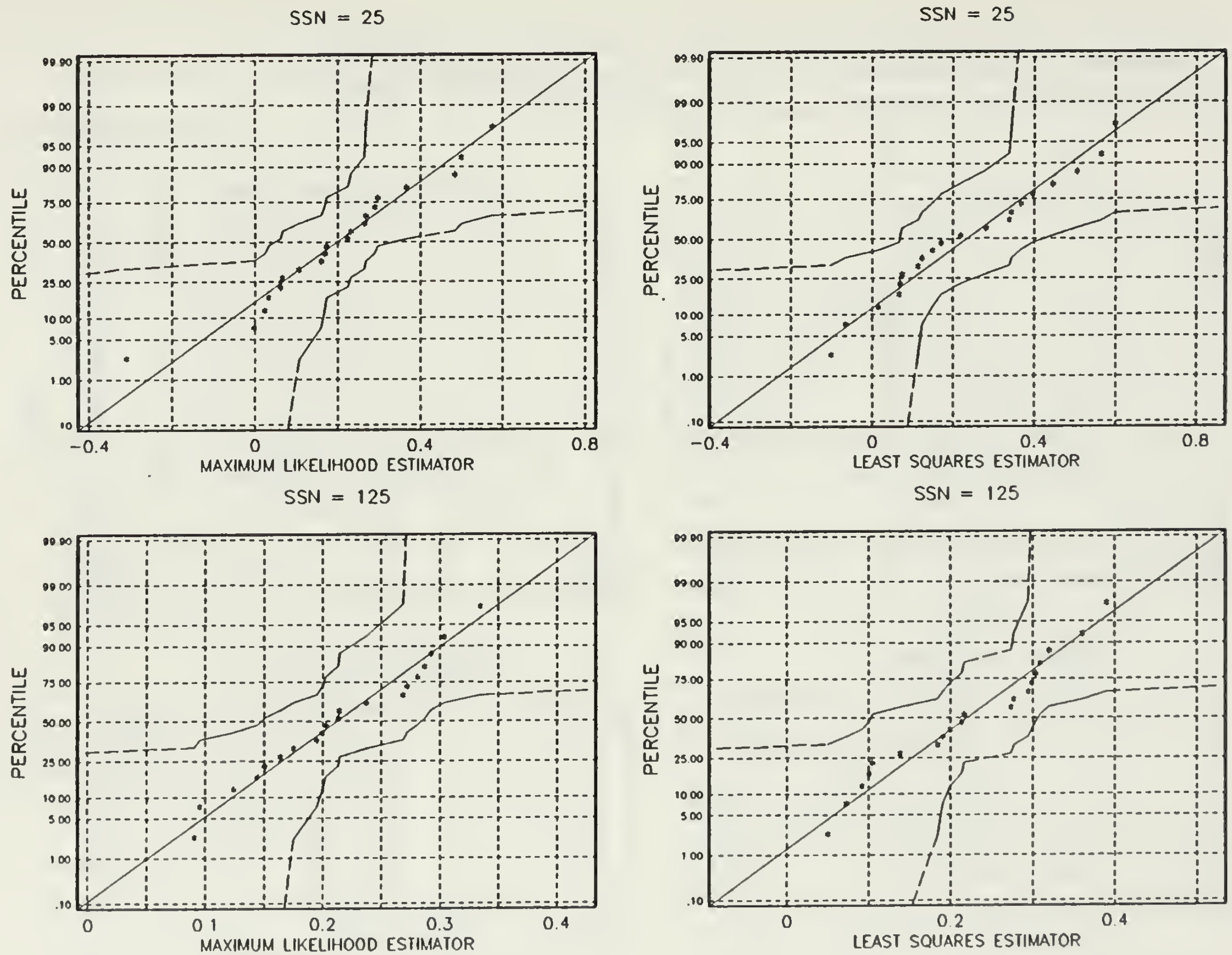


Figure III.E.6.1. Normal Probability Plots of the Maximum Likelihood and the Least Squares Estimators of γ in the BELAR(1) Process for 20 Samples of Sizes 25 and 125 with α=.11 and γ=.19216

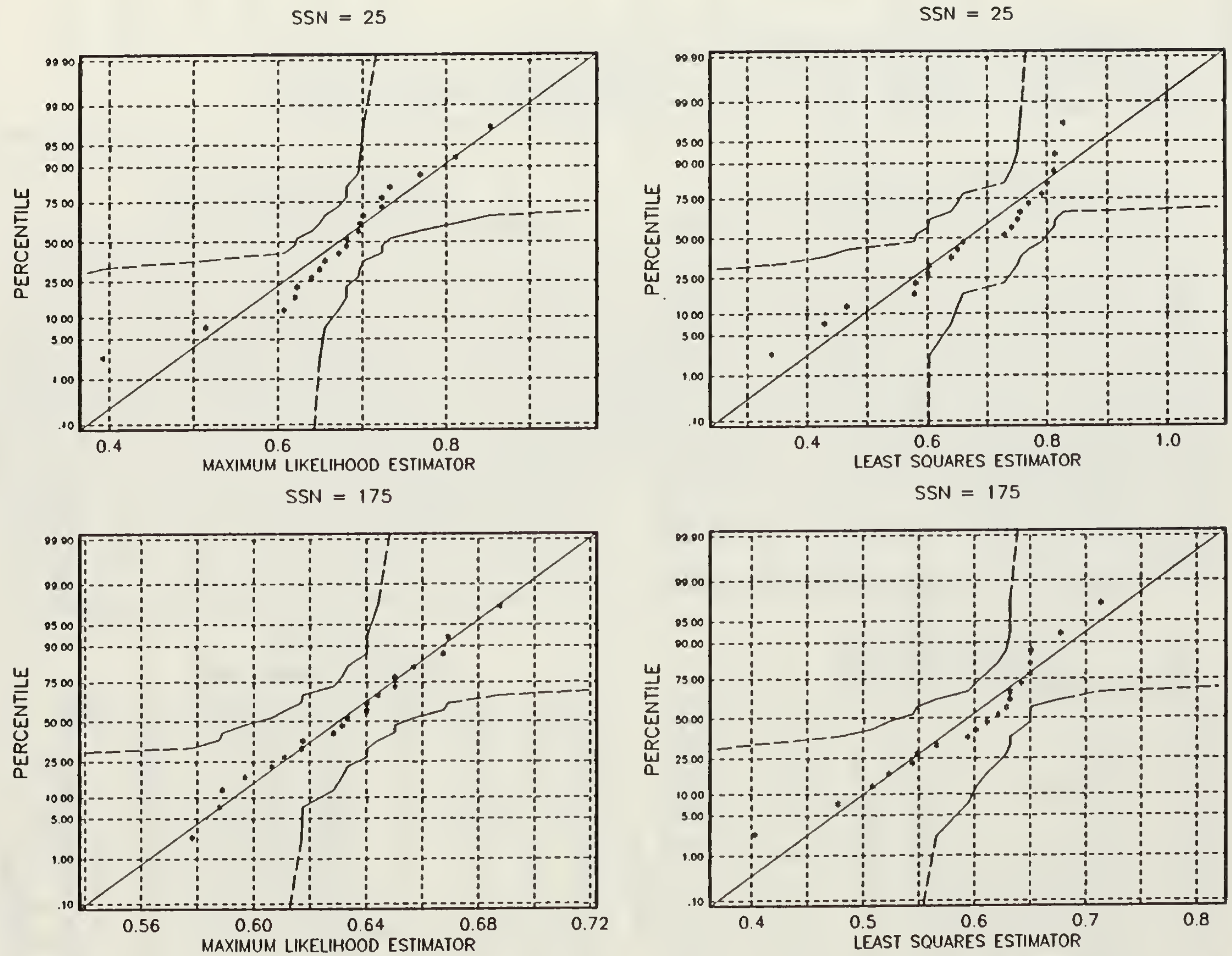


Figure III.E.6.2. Normal Probability Plots of the Maximum Likelihood and the Least Squares Estimators of γ in the BELAR(1) Process for 20 Samples if Sizes 25 and 175 with α=.5 and γ=.63662

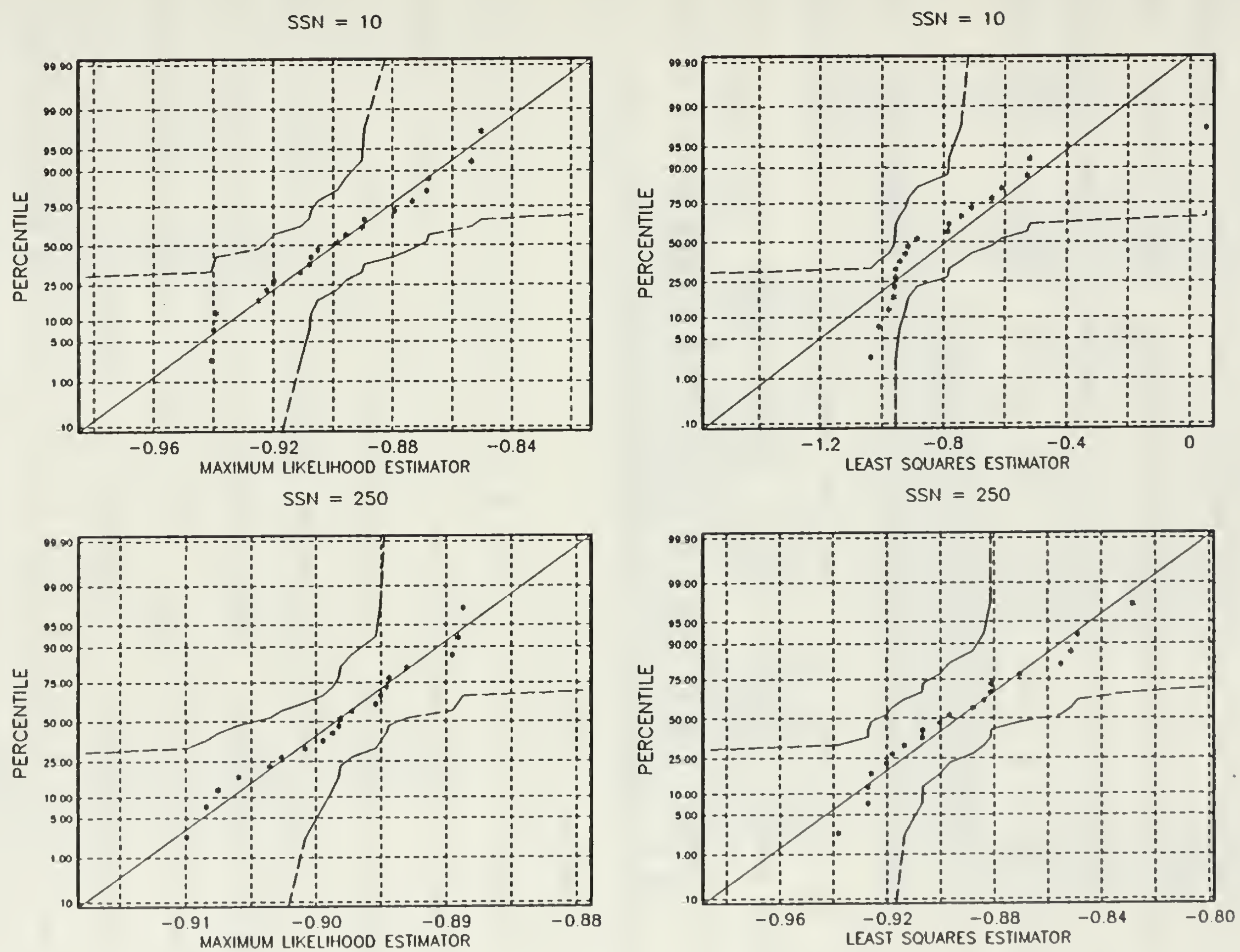
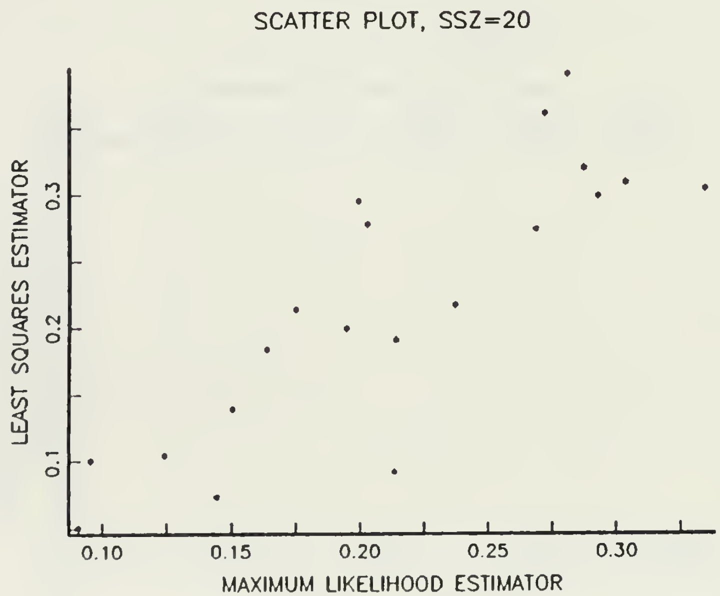


Figure III.E.6.3. Normal Probability Plots of the Maximum Likelihood and the Least Squares Estimators of γ in the BELAR(1) Process for 20 Samples of Sizes 10 and 250 with α=.844 and γ=-.89986

and for two subsample sizes, SSN. The layout provides for two-way comparisons. That is, $\hat{\gamma}_{MLE}$ from smaller SSN can be compared to $\hat{\gamma}_{MLE}$ for larger SSN. Likewise, for a given SSN, $\hat{\gamma}_{MLE}$ can be compared to $\hat{\gamma}_{LS}$, which is known to have an asymptotic Normal distribution. The straight line in the Normal plots corresponds to a Normal distribution. The curved lines correspond to the Kolmogorov-Smirnoff bounds calculated from the data at the 95% confidence level by the routine in the IBM experimental APL routine called GRAFSTAT.

It appears from these figures that for the larger values of SSN, $\hat{\gamma}_{MLE}$ and $\hat{\gamma}_{LS}$ fit Normal distributions better than the corresponding samples from the smaller values of SSN. It also appears that $\hat{\gamma}_{MLE}$ fits a Normal distribution as well as the $\hat{\gamma}_{LS}$ for the larger values of SSN. This supports the notion that the maximum likelihood estimator is converging to a Normal distribution.

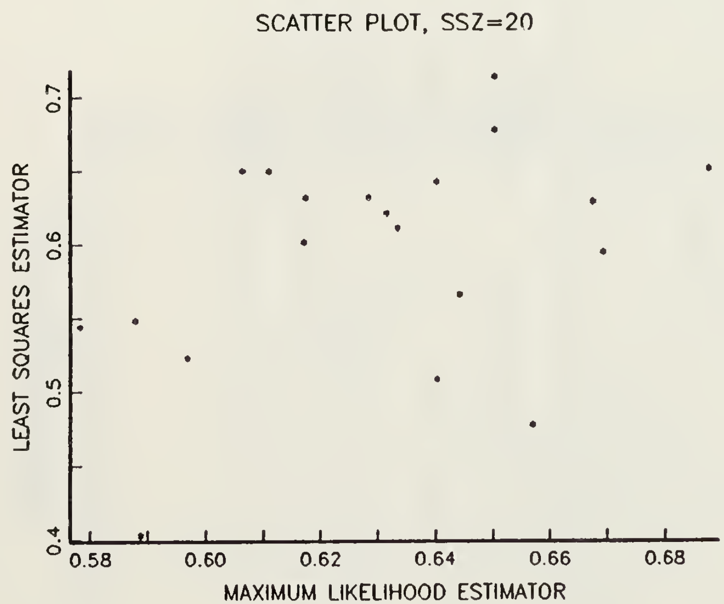
Figures III.E.6.4, III.E.6.5 and III.E.6.6 are the corresponding scatter plot analyses for the data in the previous figures for the larger value of SSN. It appears that $\hat{\gamma}_{MLE}$ and $\hat{\gamma}_{LS}$ have a positive correlation coefficient which becomes more pronounced as the data becomes less correlated. The distribution of $\hat{\gamma}_{MLE}$ also appears to have a smaller variance than $\hat{\gamma}_{LS}$. This effect is more pronounced for more highly correlated data. Compare, for example, the estimated standard deviation of $\hat{\gamma}_{MLE}$ and that of $\hat{\gamma}_{LS}$ from the table in Figure III.E.6.4 with the corresponding entries in the table from Figure III.E.6.6.



SCATTER PLOT TABLE

X	:X8
Y	:Y8
SELECTION	:ALL
X LABEL	:MAXIMUM LIKELIHOOD ESTIMATOR
Y LABEL	:LEAST SQUARES ESTIMATOR
NO. OF ELEMENTS	:20
CORRELATION XY	:0.84314
RK CORRELATION	:0.84361 I=6.6656
X MEAN	:0.21228
STD. DEVIATION	:0.069447
5-PERCENTILE	:0.09065
25-PERCENTILE	:0.15051
MEDIAN	:0.20274
75-PERCENTILE	:0.2721
95-PERCENTILE	:0.30362
X MIN.	:0.09065 0.09531 0.12411
X MAX.	:0.33468 0.30362 0.29269
Y MEAN	:0.21954
STD. DEVIATION	:0.098889
5-PERCENTILE	:0.05024
25-PERCENTILE	:0.10457
MEDIAN	:0.21345
75-PERCENTILE	:0.29888
95-PERCENTILE	:0.36103
Y MIN	:0.05024 0.0729 0.09175
Y MAX	:0.39016 0.36103 0.31999

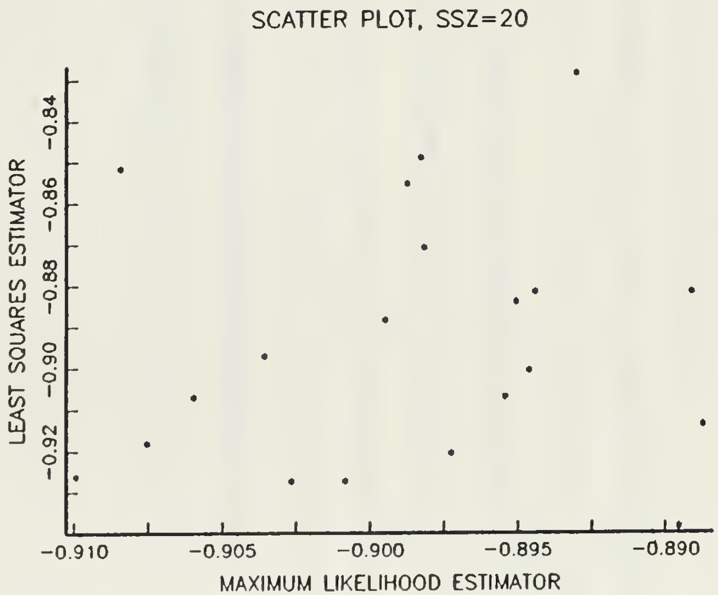
Figure III.E.6.4. Scatter Plot Analysis of the Maximum Likelihood and the Least Squares Estimators of γ in the BELAR(1) Process for 20 Samples of Size 125 with α=.11 and γ=.19216



SCATTER PLOT TABLE

X	:X1
Y	:Y1
SELECTION	:ALL
X LABEL	:MAXIMUM LIKELIHOOD ESTIMATOR
Y LABEL	:LEAST SQUARES ESTIMATOR
NO. OF ELEMENTS	:20
CORRELATION XY	:0.39304
RK CORRELATION	:0.31278 T=1.3971
X MEAN	:0.63015
STD. DEVIATION	:0.028945
5-PERCENTILE	:0.57814
25-PERCENTILE	:0.6064
MEDIAN	:0.63144
75-PERCENTILE	:0.65025
95-PERCENTILE	:0.66918
X MIN.	:0.57814 0.58779 0.58881
X MAX.	:0.6877 0.66918 0.66747
Y MEAN	:0.59386
STD. DEVIATION	:0.072804
5-PERCENTILE	:0.40305
25-PERCENTILE	:0.54442
MEDIAN	:0.61131
75-PERCENTILE	:0.64233
95-PERCENTILE	:0.6775
Y MIN	:0.40305 0.47757 0.50861
Y MAX	:0.71358 0.6775 0.65112

Figure III.E.6.5. Scatter Plot Analysis of the Maximum Likelihood and the Least Squares Estimators of γ in the BELAR(1) Process for 20 Samples of Size 175 with α=.5 and γ=.63662



SCATTER PLOT TABLE

X	:X12
Y	:Y12
SELECTION	:ALL
X LABEL	:MAXIMUM LIKELIHOOD ESTIMATOR
Y LABEL	:LEAS¯ SQUARES ESTIMATOR
NO. OF ELEMENTS	:20
CORRELATION XY	:0.10067
RK CORRELATION	:0.15038 T=0.64533
X MEAN	:¯0.89854
STD. DEVIATION	:0.0061886
5-PERCENTILE	:¯0.90994
25-PERCENTILE	:¯0.90358
MEDIAN	:¯0.89827
75-PERCENTILE	:¯0.89462
95-PERCENTILE	:¯0.8891
X MIN.	:¯0.90994 ¯0.90844 ¯0.90754
X MAX.	:¯0.88872 ¯0.8891 ¯0.88955
Y MEAN	:¯0.89354
STD. DEVIATION	:0.029929
5-PERCENTILE	:¯0.9383
25-PERCENTILE	:¯0.92028
MEDIAN	:¯0.90045
75-PERCENTILE	:¯0.88127
95-PERCENTILE	:¯0.84888
Y MIN	:¯0.9383 ¯0.92718 ¯0.92717
Y MAX	:¯0.82831 ¯0.84888 ¯0.85141

Figure III.E.6.6. Scatter Plot Analysis of the Maximum Likelihood and the Least Squares Estimators of γ in the BELAR(1) Process for 20 Samples of Size 250 with α=.844 and γ=-.89986

F. ℓ-LAPLACE MOVING AVERAGE MODELS

1. Introduction

In this section, we derive a time series model that has an ℓ-Laplace marginal distribution and the correlation structure of a linear q^{th}-order moving average model. This construction uses the square root Beta-Laplace transform given in Section III.B.3. The first-order model retains the full range of correlations up to 1/2.

2. The First-Order Moving Average Model

Let $\{L_n(\ell-\alpha)\}$ be an i.i.d. sequence of $(\ell-\alpha)$-Laplace random variables; $\{A_n^{1/2}(\alpha,\ell-2\alpha)\}$ be an i.i.d. sequence, independent of $\{L_n(\ell-\alpha)\}$, where A_n is a Beta $(\alpha,\ell-2\alpha)$ random variable and $0 < \alpha < \ell/2$. Both the innovation and the coefficient sequences are independent of X_{n-1}, X_{n-2}, ... Then the sequence $\{X_n(\ell\}$ given by

$$X_n(\ell) = L_n(\ell-\alpha) + A_n^{1/2}(\alpha,\ell-2\alpha)L_{n-1}(\ell-\alpha), \qquad \text{(III.F.2.1)}$$

has a marginal ℓ-Laplace distribution and an MA(1) correlation structure such that $0 < \mathrm{Corr}(X_n,X_{n-1}) < 1/2$.

To see that $X_n(\ell)$ has an ℓ-Laplace distribution, first note that by the square root Beta-Laplace transform theorem of Section III.B.3, the distribution of the product $A_n^{1/2}(\alpha,\ell-2\alpha)L_{n-1}(\ell-\alpha)$ is α-Laplace. Then note that $X_n(\ell)$ is the sum of two independent random variables, one of which has an $(\ell-\alpha)$-Laplace distribution and the other has an α-Laplace distribution. So, if $\phi_X(\omega)$ is the characteristic function of $X_n(\ell)$, then

$$\phi_X(\omega) = \left\{\frac{1}{1+\omega^2}\right\}^{\ell-\alpha} \left\{\frac{1}{1+\omega^2}\right\}^{\alpha} = \left\{\frac{1}{1+\omega^2}\right\}^{\ell} . \qquad \text{(III.F.2.2)}$$

To see that $\{X_n(\ell)\}$ has the correct correlation structure, first note that by the construction of (III.F.2.1), X_{n-k} is explicitly independent of X_n for $|k| \geq 2$. Therefore, $\mathrm{Corr}(X_n,X_{n-k})$ is zero for $|k| \geq 2$.

For $k = \pm 1$, we have, after some simplification

$$\mathrm{Corr}(X_n,X_{n-k}) = \frac{\alpha\Gamma(\alpha+\frac{1}{2})\Gamma(\ell+1-\alpha)}{\ell\Gamma(\alpha+1)\Gamma(\ell-\alpha+\frac{1}{2})} . \qquad \text{(III.F.2.3)}$$

Finally, note that in the limit as $\alpha \to 0$, (III.F.2.3) approaches zero. Also, as $\alpha \to \ell/2$, (III.F.2.3) approaches 1/2.

Replace $A_n^{1/2}(\alpha,\ell-2\alpha)$ in (4.1) by $-A_n^{1/2}(\alpha,\ell-2\alpha)$, we have a full range $(-1/2,0)$ of nonpositive lag-1 serial correlations.

3. The q-Order Moving Average Model

The MA(q) model for $q \geq 2$ is the extension of the MA(1) model given in (III.F.2.1). Let $\{L_n(\ell-q\alpha)\}$ be an i.i.d. sequence of $(\ell-q\alpha)$-Laplace random variables. Let $[A_{n,i}^{1/2}\{\alpha,\ell-(q+1)\alpha\}]$ for $i = 1,\ldots,q$ be i.i.d. sequences, independent of each other and of $\{L_n(\ell-q\alpha)\}$, where $A_{n,i}$ is a Beta $\{\alpha,\ell-(q+1)\alpha\}$ random variable for all n and all $i = 1,\ldots,q$. Also, $0 < \alpha < \ell/(q+1)$. Both the innovation and each of the coefficient sequences are independent of $X_{n-1},X_{n-2},\ldots$. Then the sequence $\{X_n(\ell)\}$ given by

$$X_n(\ell) = L_n(\ell-q\alpha) + \sum_{i=1}^{q} A_{n,i}^{1/2}\{\alpha,\ell-(q+1)\alpha\}L_{n-i}(\ell-q\alpha), \qquad \text{(III.F.3.1)}$$

has a marginal ℓ-Laplace distribution and an MA(q) correlation structure for $0 < \alpha < \ell/(q+1)$. When $\alpha = 0$, then $\{X_n(\ell)\}$ is an i.i.d. sequence; when $\alpha = \ell/(q+1)$, then the moving average is an equal weighted average of q+1 i.i.d. α-Laplace error terms $L_n(\alpha)$.

To see that $X_n(\ell)$ is an ℓ-Laplace random variable, first note from the square root Beta-Laplace transformation theorem of Section III.B.3, that each product $A_{n,i}^{1/2}\{\alpha, \ell-(q+1)\alpha\}L_{n-i}(\ell-q\alpha)$ has an α-Laplace distribution.

But the sum of q i.i.d. α-Laplace random variables has a $q\alpha$-Laplace distribution. Thus, $X_n(\ell)$ is the sum of two independent random variables and its characteristic function is obtained as the product

$$\phi_X(\omega) = \left\{\frac{1}{1+\omega^2}\right\}^{\ell-q\alpha} \prod_{i=1}^{q} \left\{\frac{1}{1+\omega^2}\right\}^{\alpha}$$

$$= \left\{\frac{1}{1+\omega^2}\right\}^{\ell-q\alpha} \left\{\frac{1}{1+\omega^2}\right\}^{q\alpha} = \left\{\frac{1}{1+\omega^2}\right\}^{\ell}. \qquad \text{(III.F.3.2)}$$

The correlations are truncated at lags $|k| \geqq q+1$. By the construction of (III.F.3.1), X_n is explicitly independent of X_{n-k} for $|k| \geqq q+1$.

Negative correlations are obtainable with 2^q choices for replacing or not replacing $[A_{n,i}^{1/2}\{\alpha,\ell-(q+1)\alpha\}]$ by $[-A_{n,i}^{1/2}\{\alpha,\ell=(q+1)\alpha\}]$ in (III.F.3.1).

This model can be generalized from the one-parameter case by replacing $q\alpha$ in (III.F.3.1) with α_i in each term in the sum, and replacing $L_n(\ell - q\alpha)$ by $L_n(\ell - q \sum_{i=1}^{q} \alpha_i)$.

IV. RESIDUAL ANALYSIS COMPARISON OF THE NLAR(1) AND THE BELAR(1) PROCESSES

A. INTRODUCTION

Lawrance and Lewis [Ref. 22] developed a higher-order residual analysis for non-linear time series with autoregressive correlation structures. Specifically, they developed a third-order analysis based on the cross-correlation of the linear residual, R_n, and its square at lag k, R^2_{n-k}. They applied the analysis to the problem of modelling wind speed data. It is important to note that this analysis was done in conjunction with, and not in place of, the usual second-order analysis. As has been already pointed out, second-order analysis is sufficient for modelling only when the processes are both linear and Normal.

The residual analysis involves only joint moments of order three. In Chapter II of this thesis, it was shown that for the NLAR(p) models with p = 1,2, all the third-order moments--that is, those of the form $E(X_i X_j X_k)$ for all i, j, k--are zero. Therefore, the Lawrance and Lewis residual analysis will not differentiate between the NLAR(p) processes with the same autocorrelation structure. It can also be shown by induction on k that $\mathrm{Corr}(R_n, R^2_{n-k}) = \mathrm{Corr}(X^2_n, R_{n-k}) = 0$ in the BELAR(1) process. Hence, either third-order residual analysis will be unable to discriminate the BELAR(1) process from any of the NLAR(1) processes with the same autocorrelation function.

In the spirit of looking at the lowest possible joint moments for differentiating between models with symmetric marginals, a fourth-order

analysis is proposed. Two candidates are investigated as the basis of this analysis. The first one considered is the cross-correlation of X_n^3 and the linear residual at lag k, R_{n-k}. The second is the autocorrelation of R_n^2 and R_{n-k}^2. The two analyses are compared by their abilities to differentiate among the different types of NLAR(1) processes and the BELAR(1) process.

Table IV.A.1. contains a summary of the models in the comparison, along with the selected sets of parameter values and corresponding correlation coefficient. Even though each of the models has the same marginal distribution (standard Laplace) and identical autocorrelation functions, each has a theoretical cross-correlation function in terms of (X_n^3, R_{n-k}) and autocorrelation function for (R_n^2, R_{n-k}^2) that are different.

The question of how the residual analysis is affected by parameter estimation is an important issue, but is not addressed at this time.

Before the candidates are developed in the next two sections, it is convenient now to place both the NLAR(1) and BELAR(1) processes into their common RCA(1) framework.

Using the terminology of Nicholls and Quinn [Ref. 16], both the NLAR(1) and the BELAR(1) processes can be written as

$$X_n = cX_{n-1} + B_n X_{n-1} + \varepsilon_n, \qquad \text{(IV.A.1)}$$

where $\{\varepsilon_n\}$ is the i.i.d. innovation, $E(\varepsilon_n) = 0$, and otherwise defined as

1. $(1-\alpha)$-Laplace in the BELAR(1) process;

2. standard Laplace, but with a degenerate component at the origin in the LAR(1) process;

3. scaled Laplace where $\lambda = \sqrt{1-\alpha_1}$ in the TLAR(1) process;

4. convex mixture of scaled Laplace variables in the general non-boundary NLAR(1) process.

TABLE IV.A.1

Summary of Models with Laplace Marginals and Autocorrelations of $\gamma^{|k|}$

Model	Parameter Values	γ	Comments
	$\alpha_1 = 1;\ \beta_1 = .19216$	.19216	Linear models;
LAR(1)	$\alpha_1 = 1;\ \beta_1 = -.63662$	-.63662	one boundary of
	$\alpha_1 = 1;\ \beta_1 = .89986$	.89986	NLAR(1) family.
	$\alpha_1 = \beta_1 = .43836$	.19216	General discrete
NLAR(1)	$\alpha_1 = .797885;\ \beta_1 = -\alpha_1$	-.63662	random coefficient
	$\alpha_1 = \beta_1 = .94861$	.89986	model.
	$\alpha = .11$; positive model	.19216	General continuous
BELAR(1)	$\alpha = .50$; negative model	-.63662	random coefficient
	$\alpha = .884$; positive model	.89986	model.
	$\alpha_1 = .19216;\ \beta_1 = 1$	.19216	Other boundary
TLAR(1)	$\alpha_1 = .63662;\ \beta_1 = -1$	-.63662	model of NLAR(1).
	$\alpha_1 = .89986;\ \beta_1 = 1$	.89986	

The $\{B_n\}$ process is the i.i.d. random coefficient process, independent of $\{\varepsilon_n\}$ and $\{X_k\}$ for $k \leq n-1$ with $E(B_n) = 0$ and otherwise defined as:

1. $\pm(A_n^{1/2}(\alpha,1-\alpha) - \gamma)$, where $\gamma = E\{A_n^{1/2}(\alpha,1-\alpha)\}$ and $A_n(\alpha,1-\alpha)$ is a standard Beta random variable in the BELAR(1) process;
2. 0 in the LAR(1) process, since it is a linear, constant coefficient AR(1) process;
3. $\beta_1\{K_n'-\alpha\}$ in the other NLAR(1) processes, where K_n' is a Bernoulli random variable such that $P(K_n' = 1) = \alpha_1$ and $0 \leq |\beta_1| \leq 1$ and α_1 and β_1 are not both unity. At $\beta_1 = \pm 1$ the process is the TLAR(1) process.

The c is a constant defined as:

1. $\gamma = E\{A_n^{1/2}(\alpha,1-\alpha)\}$ in the BELAR(1) process;
2. $\alpha_1\beta_1 = \beta_1 E(K_n')$ in all the NLAR(1) processes ($\alpha_1 = 1$ being the LAR(1) process).

The second and fourth moments of E_n and the second, third and fourth moments of B_n are needed in the following sections. In Table IV.A.2, there is a convenient summary of the necessary equations.

Now the linear residual, written in terms from (IV.A.1) has the following forms analogous to (III.E.4.3) and (III.E.4.4),

$$R_n = B_n X_{n-1} + \varepsilon_n, \tag{IV.A.2}$$

$$R_n = X_n - cX_{n-1}. \tag{IV.A.3}$$

Using (IV.A.2) and the independence of $\{B_n\}$ and $\{\varepsilon_n\}$, the second and fourth moments of R_n are

TABLE IV.A.2

Various Moments for B_n and ε_n in the RCA(1) Models

Moments	LAR(1)	NLAR(1)	BELAR(1)	TLAR(1)
$E(B_n^2)$	0	$\beta_1^2\alpha_1(1-\alpha_1)$	$\alpha-\gamma^2$	$\alpha_1(1-\alpha_1)$
$E(B_n^3)$	0	$\beta_1^3\alpha_1(1-\alpha_1)(1-2\alpha_1)$	$\frac{\gamma}{3}(6\gamma^2-7\alpha+1)$	$\alpha_1(1-\alpha_1)(1-2\alpha_1)$
$E(B_n^4)$	0	$\beta_1^4\alpha_1(1-\alpha_1)(1-3\alpha_1+3\alpha_1^2)$	$\frac{\alpha(1+\alpha)}{2}+\frac{10}{3}\alpha\gamma^2-\frac{\gamma^2}{3}(4+9\gamma^2)$	$\alpha_1(1-\alpha_1)(1-3\alpha_1+3\alpha_1^2)$
$E(\varepsilon_n^2)$	$2(1-\beta_1^2)$	$2(1-\alpha_1\beta_1^2)$	$2(1-\alpha)$	$2(1-\alpha_1)$
$E(\varepsilon_n^4)$	$24(1-\beta_1^2)$	$24[1-\alpha_1\beta_1^2\{1+(1-\alpha_1)\beta_1^2\}]$	$12(1-\alpha)(2-\alpha)$	$24(1-\alpha_1)^2$

$$E(R_n^2) = 2E(B_n^2) + E(\varepsilon_n^2), \qquad \text{(IV.A.4)}$$

$$E(R_n^4) = 24E(B_n^4) + 12E(B_n^2)E(\varepsilon_n^2) + E(\varepsilon_n^4). \qquad \text{(IV.A.5)}$$

The variance of R_n^2 when needed is derived from (IV.A.4) and IV.A.5).

B. RESIDUAL ANALYSIS USING $\mathrm{Corr}(X^3{}_n, R_{n-k})$

In this section, the residual analysis using the theoretical cross-correlations, $\mathrm{Corr}(X_n^3, R_{n-k})$ is developed. Using (IV.A.1) and (IV.A.2), we have

$$X_n^3 = c^3X_{n-1}^3 + 3c^2X_{n-1}^2R_n + 3cX_{n-1}R_n^2 + R_n^3, \qquad \text{(IV.B.1)}$$

where c is defined in Section IV.A.

The cross-correlation function of X_n^3 and R_{n-k} at lag k is defined as

$$C_{31}(k) = \mathrm{Corr}(X_n^3, R_{n-k}) = \frac{E(X_n^3R_{n-k})}{\sigma_{X_n^3}\sigma_{R_{n-k}}}, \qquad \text{(IV.B.2)}$$

where $\mathrm{Var}(X_n^3) = E(X_n^6) = 6!$ and $\mathrm{Var}(R_{n-k})$ is given by (IV.A.4) for all n and all k, since as shown in Section III.E.3, $\{R_n\}$ is stationary whenever $\{X_n\}$ is.

Now from the construction of R_n in (IV.A.2), it is explicity clear that X_n and R_{n-k} are dependent for all k and that the $\{R_n\}$ are not independent of each other, unless B_n is identically zero as in linear

constant coefficient AR(1) processes. However, by the Residual Theorem (Lawrance and Lewis [Ref. 22]), the $\{R_n\}$ are uncorrelated.

From (IV.B.1), we have for all k

$$C_{31}(k) = \{c^3E(X_{n-1}^3R_{n-k}) + 3c^2E(X_{n-1}^2R_nR_{n-k}) + 3cE(X_{n-1}R_n^2R_{n-k}) + E(R_n^3R_{n-k})\} / [12\sqrt{5}\,\{E(R_n^2)\}^{1/2}]. \qquad \text{(IV.B.3)}$$

Consider (IV.B.3) for $k < 0$. Since the random coefficients $\{B_n\}$ are independent of the process $\{X_j\}$ for $j \leqq n-1$, this implies that $C_{31}(k)$ is zero for $k < 0$. For $k = 0$ in (IV.B.3), we have, after some simplification,

$$C_{31}(0) = \frac{72c^2E(B_n^2) + 6c^2E(\varepsilon_n^2) + 72cE(B_n^3) + E(R_n^4)}{12\sqrt{5}\,\{E(R_n^2)\}^{1/2}}. \qquad \text{(IV.B.4)}$$

For $k \geqq 1$, there is the following recursive formula,

$$C_{31}(k) = C_{31}(K-1)\{c^3 + 3cE(B_n^2) + E(B_n^3)\} + \frac{c^k(1-c^2)E(\varepsilon_n^2)}{2\sqrt{5}\,\{E(R_n^2)\}^{1/2}}. \qquad \text{(IV.B.5)}$$

It is now a simple matter to consolidate the expressions for $C_{31}(k)$ for all k and substitute the appropriate expressions from Table IV.A.2. For the NLAR(1) models--including LAR(1), for which $\alpha_1 = 1$, and TLAR(1) for which $\beta_1 = \pm 1$--we have

$$C_{31}(k) = \begin{cases} 0, & k < 0; \\ \dfrac{\{2 - \alpha_1^2\beta_1^2 + 6\alpha_1\beta_1^3(1-2\alpha_1)(1-\alpha_1) - \alpha_1^2\beta_1^4(8-19\alpha_1+12\alpha_1^2)\}}{\sqrt{10}\,(1-\alpha_1^2\beta_1^2)^{1/2}}, & k = 0; \\ \alpha_1\beta_1^3 C_{31}(k-1) + \dfrac{\alpha_1^k\beta_1^k(1-\alpha_1\beta_1^2)(1-\alpha_1\beta_1^2)^{1/2}}{\sqrt{10}}, & k \geqq 1. \end{cases} \qquad \text{(IV.B.6)}$$

For the BELAR(1) process, we have

$$C_{31}(k) = \begin{cases} 0, & k < 0; \\ \dfrac{(6 - 5\gamma^2 - \alpha\gamma^2)}{3\sqrt{10}\,(1-\gamma^2)^{1/2}}, & k = 0; \\ \dfrac{\gamma}{3}(1+2\alpha)C_{31}(k-1) + \dfrac{\gamma^k(1-\alpha)(1-\gamma^2)^{1/2}}{\sqrt{10}}, & k \geqq 1. \end{cases} \qquad \text{(IV.B.7)}$$

The theoretical cross correlation functions for each of the models and sets of parameters in Table IV.A.1 are given in Figures IV.B.1 - IV.B.3. Three points can be made. For the models with $|\rho|$ small, such as in Figure IV.B.1, there is little difference between the cross-correlation functions of all four models. (Of course for $\rho = 0$, there is absolutely no difference, since all NLAR(1) models and the BELAR(1) model collapse into the unique i.i.d. case). A difference between the cross-correlation function of the boundary NLAR(1) models--LAR(1) and TLAR(1)--does become more apparent as $|\rho|$ increases. But, there seems

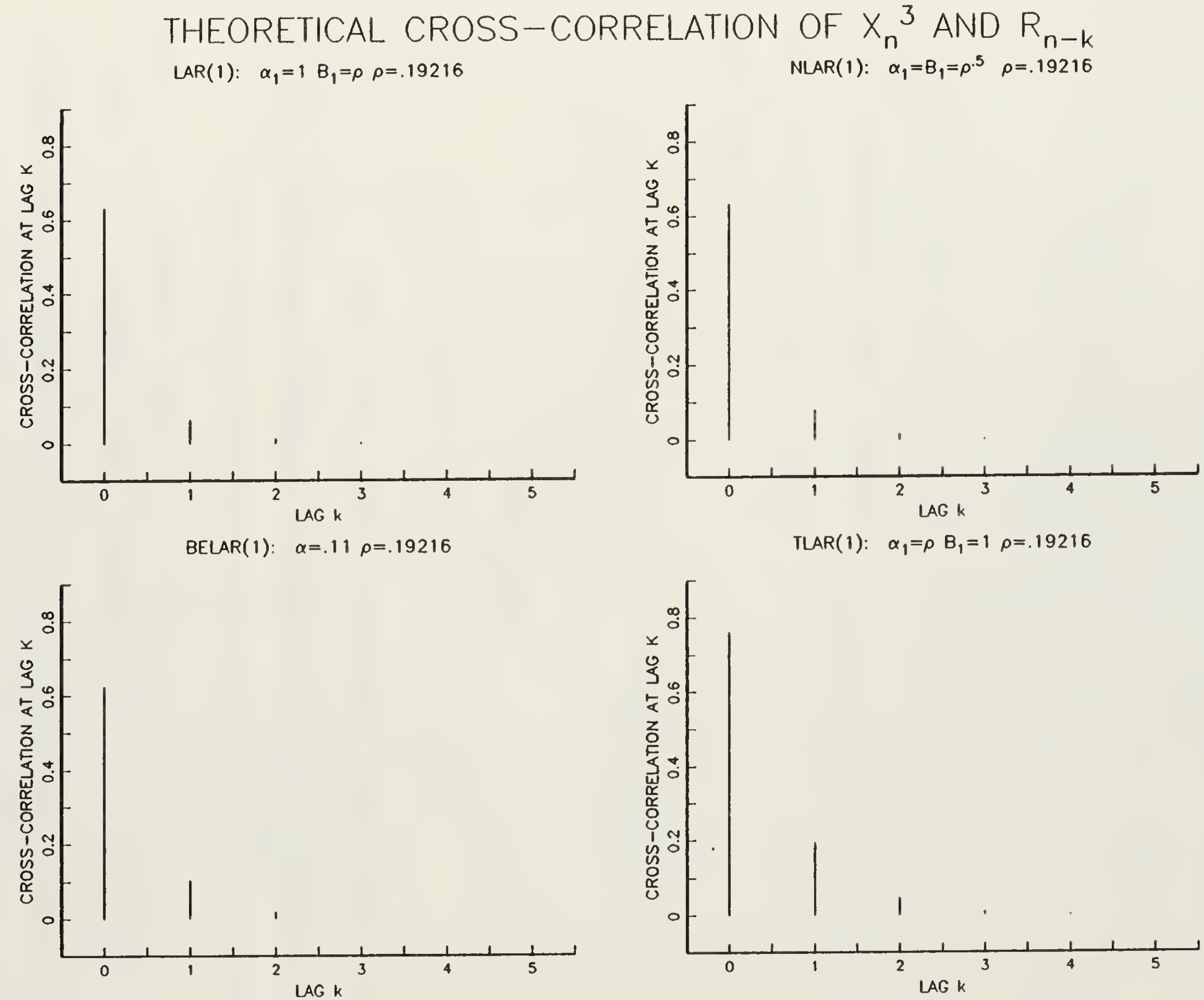


Figure IV.B.1. Theoretical Cross-Correlation Functions of X_n^3 and R_{n-k} for 4 RCA(1) Processes with $\rho(1)=.19216$

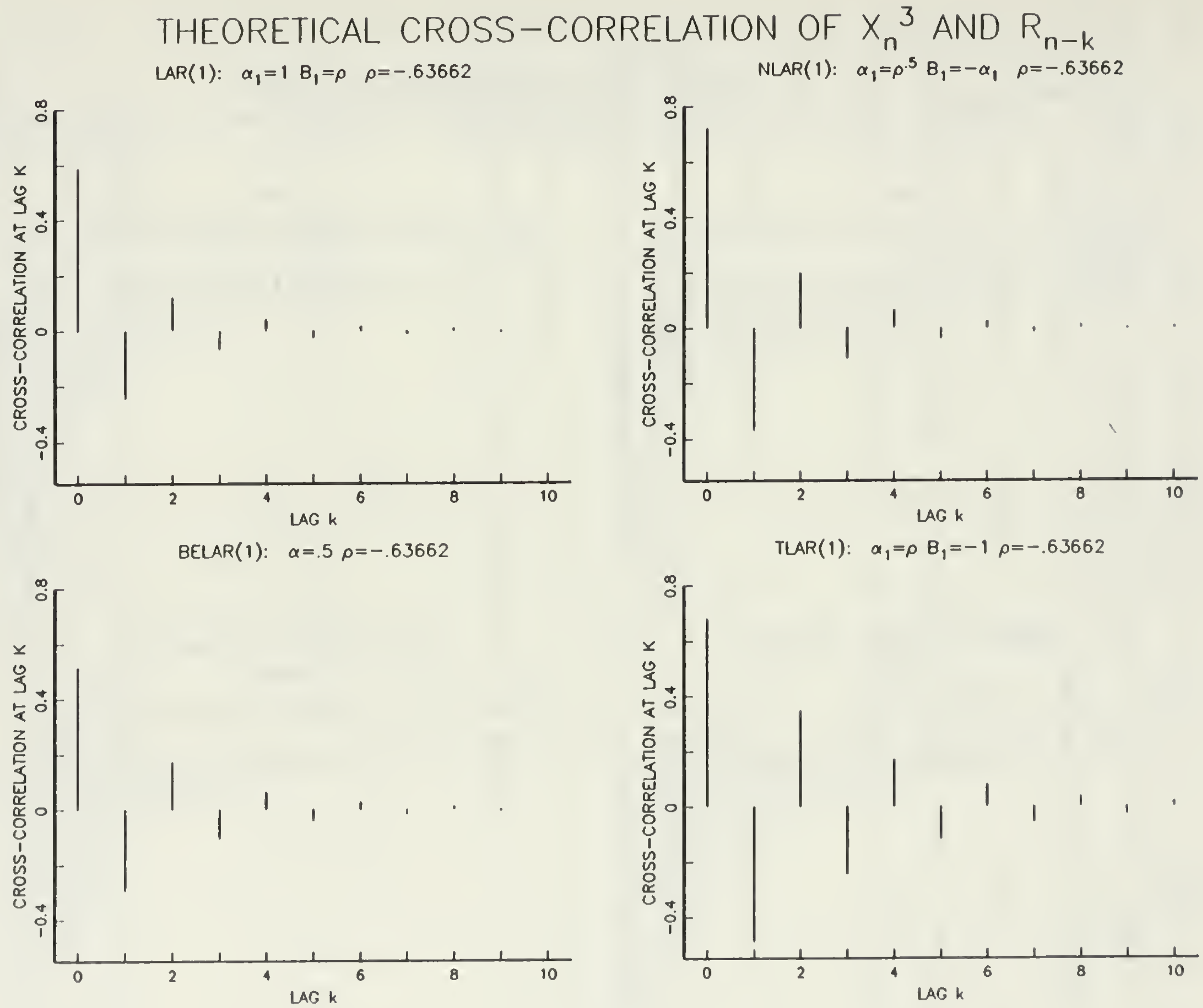


Figure IV.B.2. Theoretical Cross-Correlation Functions of X_n^3 and R_{n-k} for 4 RCA(1) Processes with $\rho(1)=-.63662$

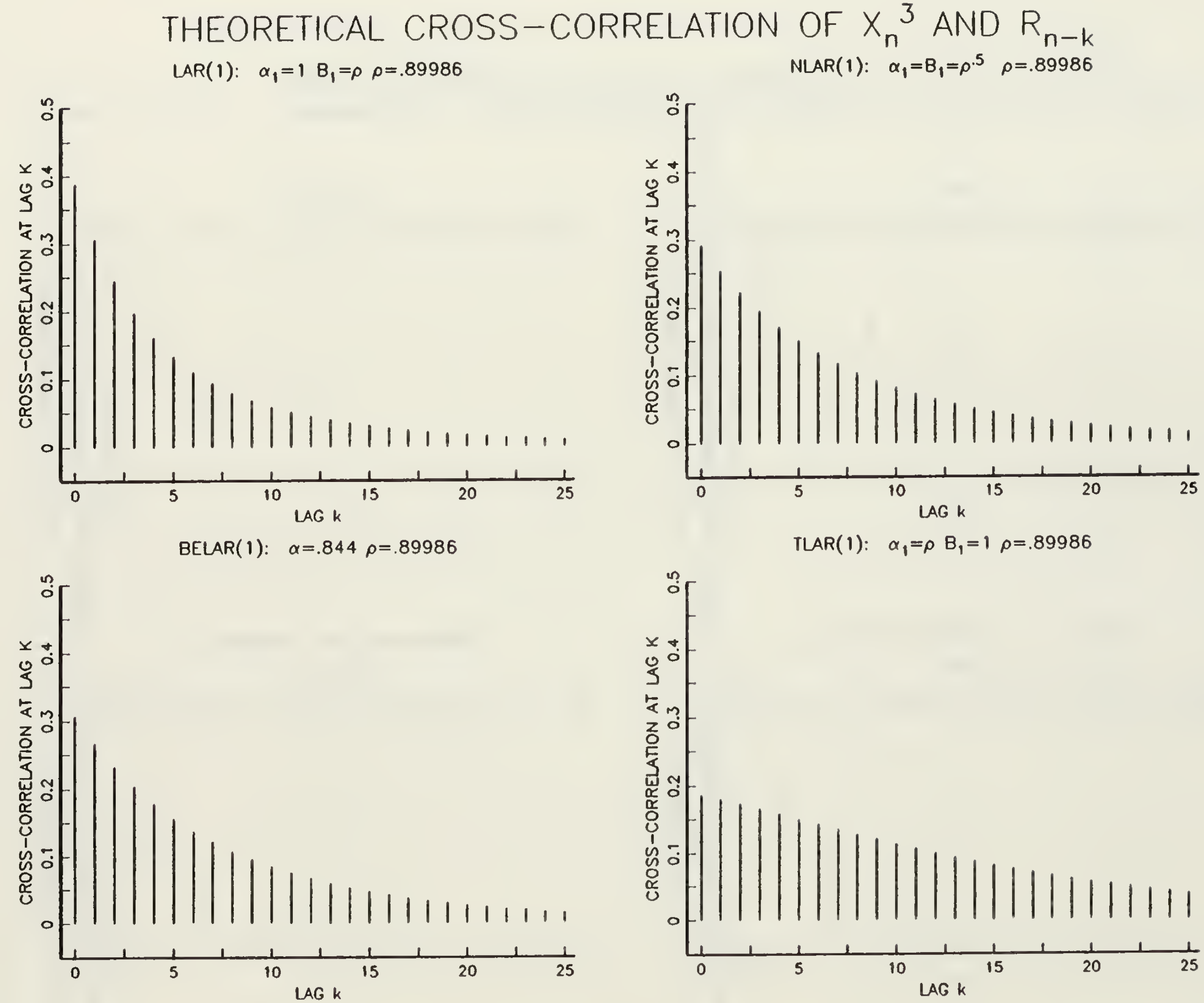


Figure IV.B.3. Theoretical Cross-Correlation Functions of X_n^3 and R_{n-k} for 4 RCA(1) Processes with $\rho(1)=.89986$

to be little distinction between the cross-correlation functions of X_n^3 and R_{n-k} from the BELAR(1) process and the non-boundary NLAR(1) process with $\alpha_1 = \beta_1 = \sqrt{|\rho|}$ even when $|\rho|$ is large as in Figure IV.B.3. This final point suggests the possibility that there exists a pair of values, (α_1, β_1), whose product is $\rho \neq 0$, for which the BELAR(1) process and the corresponding NLAR(1) process will not only have identical autocorrelation functions, but may also have nearly identical cross-correlations of X_n^3 and R_{n-k} for some specified number lags $k = 0, 1, \ldots, j$.

The cross-correlations of X_n^3 and R_{n-k} can also be used to distinguish the random coefficient AR(1) processes with a standard Laplace marginal distribution from the Gaussian AR(1) process where $X_n \sim N(0,2)$ and $\varepsilon_n \sim N\{0, 2(1-a^2)\}$, where a is the correlation coefficient. We have for the Gaussian AR(1) models,

$$C_{31}(k) = \mathrm{Corr}(X_n^3, R_{n-k}) = \begin{cases} 0 & k \leqq -1, \\ (3/5(1-a^2))^{1/2} & k = 0, \\ a^k C_{31}(0) & k \geqq 1. \end{cases} \qquad \text{(IV.B.8)}$$

Note that is $C_{31}(k)$ for $k \geqq 1$ is proportional to $\mathrm{Corr}(X_n, X_{n-k}) = a^k$. This is consistent with the fact that a Gaussian process is completely determined by the mean and covariance matrix.

Figures IV.B.4 - IV.B.6 show the theoretical cross-correlation function of the Gaussian AR(1) model superimposed on the values for the different models from Figures IV.B.1 - IV.B.3, which have the standard

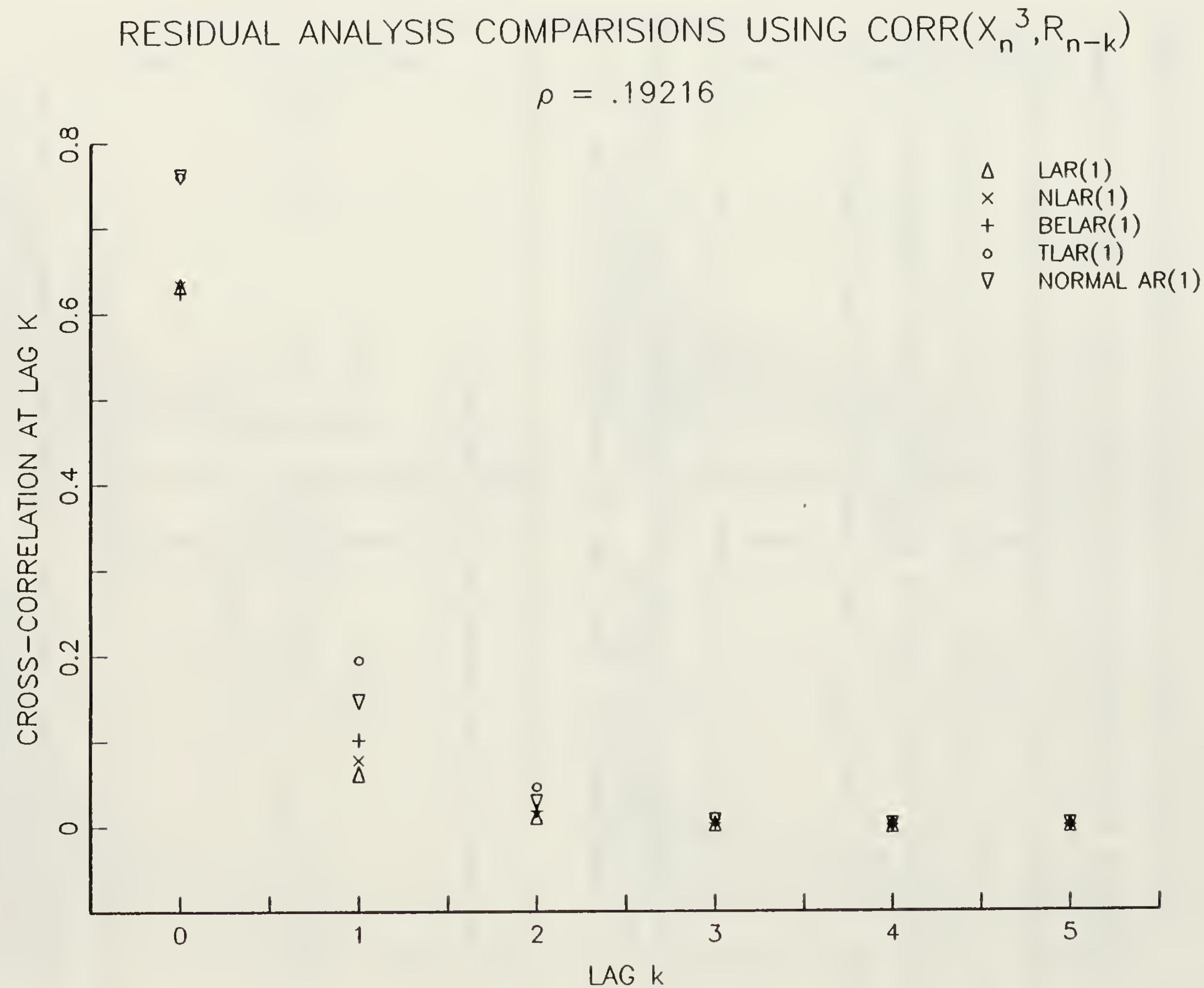


Figure IV.B.4. Residual Analysis Comparisions Using Corr(X_n^3,R_{n-k}) for the Gaussian AR(1) Process and the 4 RCA(1) Processes with Var(X_n)=2, E(X_n)=0, and ρ(1)=.19216

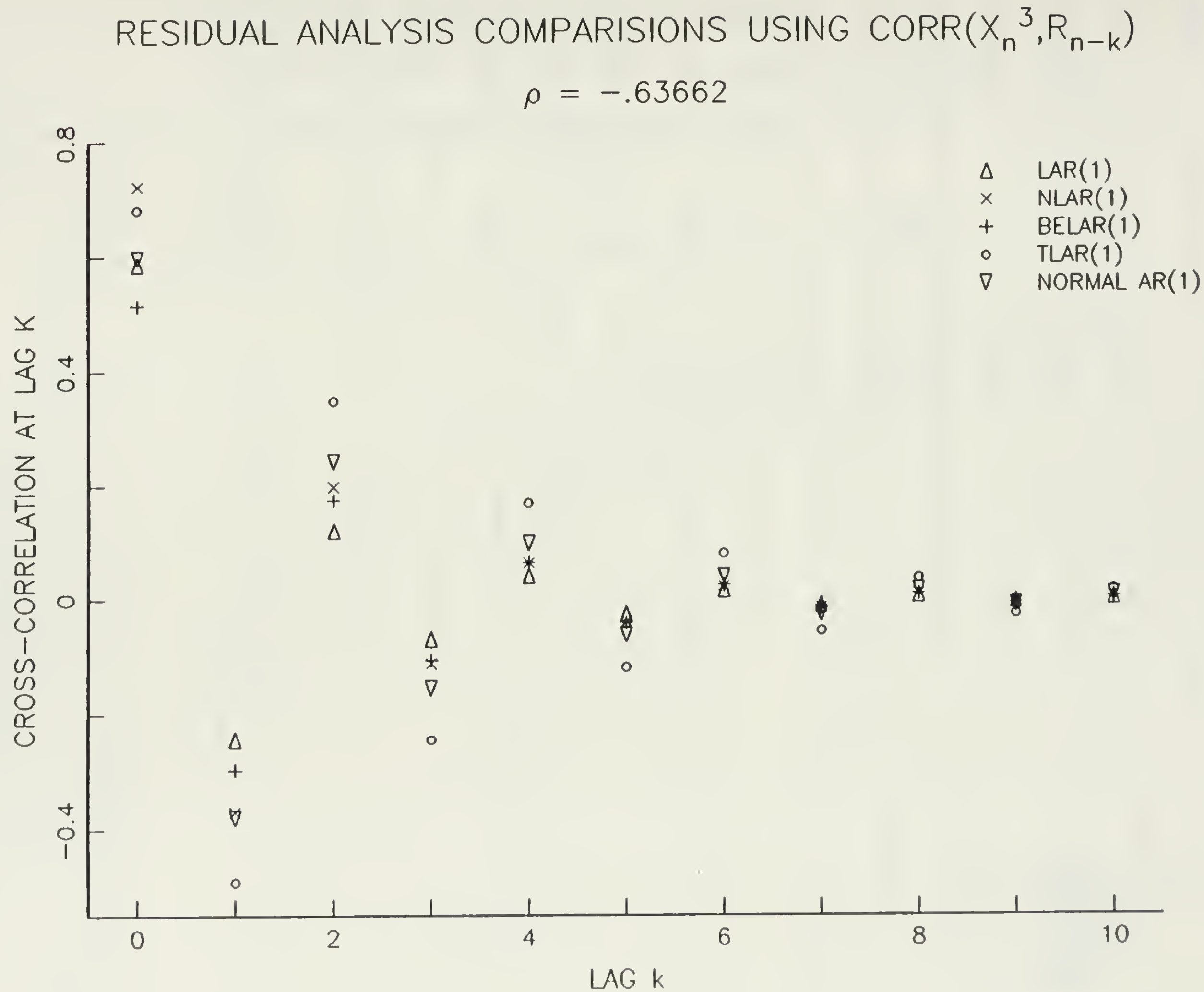


Figure IV.B.5. Residual Analysis Comparisions Using Corr(X_n^3, R_{n-k}) for the Gaussian AR(1) Process and the 4 RCA(1) Processes with $Var(X_n)=2$, $E(X_n)=0$, and $\rho(1)=-.63662$

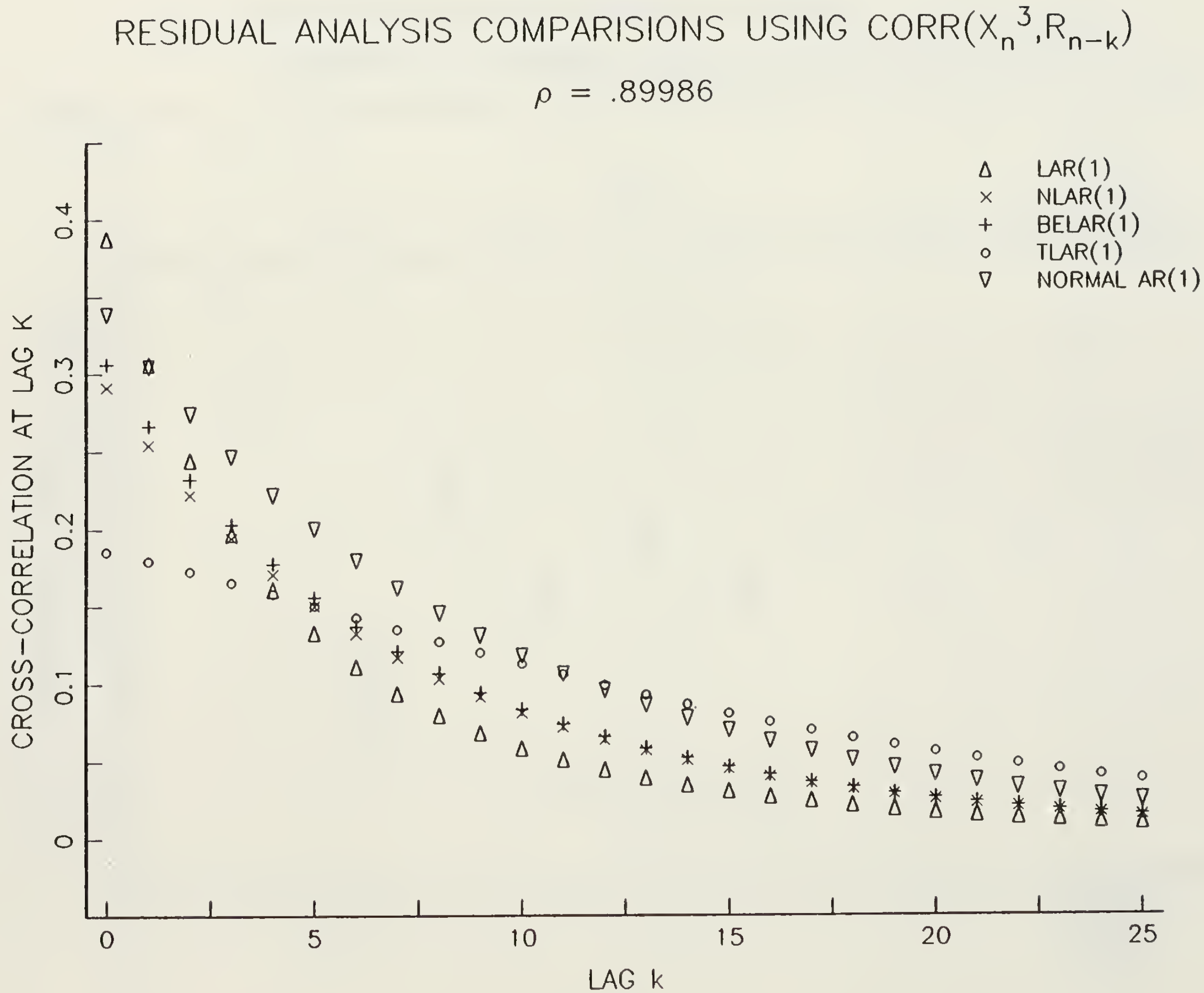


Figure IV.B.6. Residual Analysis Comparisions Using Corr(X_n^3,R_{n-k}) for the Gaussian AR(1) Process and the 4 RCA(1) Processes with Var(X_n)=2, E(X_n)=0, and $\rho(1)$=.89986

Laplace marginal distribution. There is some differentiation between the Laplace models with AR(1) correlation structure and the given Gaussian AR(1) model, but not much. It would, however, be very easy to identify the Gaussian model from the Laplace models using probability plots. This illustrates the point made at the beginning of this chapter, that a higher-order residual analysis is not intended to replace the existing methods of analysis. It also emphasizes one of the very foundations of the thesis, that the marginal distribution is one of the very first aspects of a time series that should be examined.

C. RESIDUAL ANALYSIS USING $\mathrm{Corr}(R_n^2, R_{n-k}^2)$

In this section, the residual analysis using the theoretical autocorrelations, $\mathrm{Corr}(R_n^2, R_{n-k}^2)$ is developed.

Let $C_{22}(k)$ represent $\mathrm{Corr}(R_n^2, R_{n-k}^2)$ for all k. Since the correlation function is an even function and $\{R_n\}$ is stationary, $C_{22}(k) = C_{22}(-k)$. We represent only $k = 0,1,2,\ldots$. Using (IV.A.2), we have after some simplification for $k \geq 1$,

$$C_{22}(k) = \{E(R_n^2 R_{n-k}^2) - E(R_n^2)E(R_{n-k}^2)\} / (\sigma_{R_n^2}\sigma_{R_{n-k}^2})$$

$$= [E\{(B_n^2 X_{n-1}^2 + 2B_n X_{n-1}\varepsilon_n + \varepsilon_n^2)R_{n-k}^2\} - E(R_n^2)E(R_{n-k}^2)] / (\sigma_{R_n^2}\sigma_{R_{n-k}^2})$$

$$= [E(B_n^2)E(X_{n-1}^2 R_{n-k}^2) + E(R_{n-k}^2)\{E(\varepsilon_n^2) - E(R_n^2)\} / (\sigma_{R_n^2}\sigma_{R_{n-k}^2})$$

$$= \{E(B_n^2)\ \mathrm{Cov}(X_{n-1}^2, R_{n-k}^2)\} / (\sigma_{R_n^2}\sigma_{R_{n-k}^2})$$

$$= E(B_n^2)\ \mathrm{Cov}(X_n^2, R_{n-(k-1)}^2)\ /\ \mathrm{Var}(R_n^2). \qquad \text{(IV.C.1)}$$

Now an immediate advantage to the analysis based on (IV.C.1) as opposed to that based on $\mathrm{Corr}(X_n^3, R_{n-k})$ is apparent. For the constant coefficient models, LAR(1), $C_{22}(k)$ will have a spike at lag-0 and be zero for all other lags, since $B_n = 0$. This will not be the case for the other NLAR(1) random coefficient processes or in the BELAR(1) process. It will not, however, distinguish the LAR(1) process from any linear AR(1) process. This, however, can be achieved using probability plots as mentioned previously.

To derive a computational formula from (IV.C.1.) in terms of the parameters of the process, first let $E_{22}(k) = E(X_n^2 R_{n-k}^2)$. Then, substituting in (IV.A.1) and (IV.A.2), we have, after some simplification for k = 0,

$$\begin{aligned} E_{22}(0) &= E\{(cX_{n-1} + B_n X_{n-1} + \varepsilon_n)^2 (B_n X_{n-1} + \varepsilon_n)^2\} \\ &= E(\varepsilon_n^4) + 2cE(\varepsilon_n^2) + 12E(\varepsilon_n^2)E(B_n^2) + 24c^2E(B_n^2) \\ &+ 48cE(B_n^3) + 24E(B_n^4). \end{aligned} \qquad \text{(IV.C.2)}$$

For $k \geqq 1$, we have the recursion

$$E_{22}(k) = \{c^2 + E(B_n^2)\}E_{22}(k-1) + E(\varepsilon_n^2)E(R_{n-k}^2). \qquad \text{(IV.C.3)}$$

Again using the stationarity of $\{X_n\}$ and $\{R_n^2\}$, we have the following expression for the autocorrelation function

$$C_{22}(k) = \begin{cases} 1, & k = 0; \\ \dfrac{E(B_n^2)}{Var(R_n^2)} \{E_{22}(k-1) - 2E(R_n^2)\}, & k \geq 1. \end{cases} \qquad (IV.C.4)$$

For the non-LAR(1) cases of the NLAR(1) process, we substitute from Table IV.A.2 and equations (IV.A.4) and (IV.A.5) to obtain

$$E_{22}(k) = \begin{cases} 4\{6 - \alpha_1^2\beta_1^2(5+12\beta_1^2-11\alpha_1\beta_1^2)\}, & k = 0; \\ \alpha_1\beta_1^2E_{22}(k-1) + 4(1-\alpha_1\beta_1^2)(1-\alpha_1^2\beta_1^2), & k \geq 1. \end{cases} \qquad (IV.C.5)$$

$$C_{22}(k) = \begin{cases} 1, & k = 0; \\ \dfrac{\alpha_1(1-\alpha_1)\beta_1^2\{E_{22}(k-1) - 4(1-\alpha_1^2\beta_1^2)\}}{4\{5-\alpha_1^2\beta_1^2(4 + 24\beta_1^2 - 42\alpha_1\beta_1^2 + 19\alpha_1^2\beta_1^2}; & k \geq 1. \end{cases} \qquad (IV.C.6)$$

The corresponding results for the BELAR(1) process are

$$E_{22}(k) = \begin{cases} 12(2+\alpha\gamma^2-3\gamma^2), & k = 0; \\ \alpha E_{22}(k-1) + 4(1-\alpha)(1-\gamma^2), & k \geq 1. \end{cases} \qquad (IV.C.7)$$

$$C_{22}(k) = \begin{cases} 1, & k = 0; \\ \dfrac{(\alpha-\gamma^2)\{E_{22}(k-1) - 4(1-\gamma^2)\}}{4(5-12\gamma^2+26\alpha\gamma^2-19\gamma^4)}, & k \geq 1. \end{cases} \qquad (IV.C.8)$$

The theoretical autocorrelation functions for each of the models and sets of parameters in Table IV.A.1 are given in Figures IV.C.1-3. There appears to be more discrimination between the TLAR(1) model and the other random coefficient models with $Corr(R_n^2, R_{n-k}^2)$ than with $Corr(X_n^3, R_{n-k})$, even when $|\rho|$ is small, as seen in comparing Figures

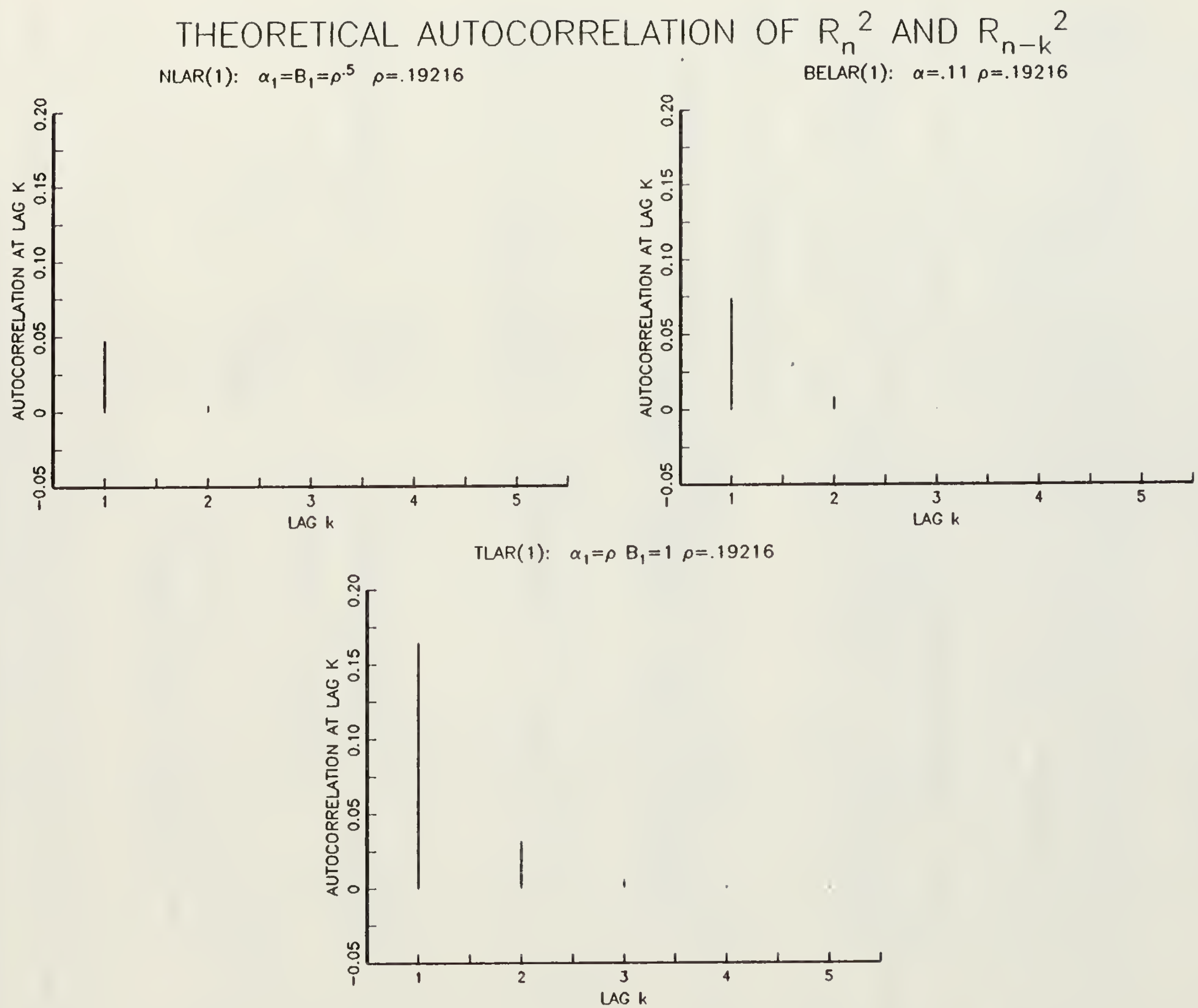


Figure IV.C.1. Theoretical Autocorrelation Functions of R_n^2 and R_{n-k}^2 for $k \geq 1$ for 3 RCA(1) Processes with $\rho(1)=.19216$

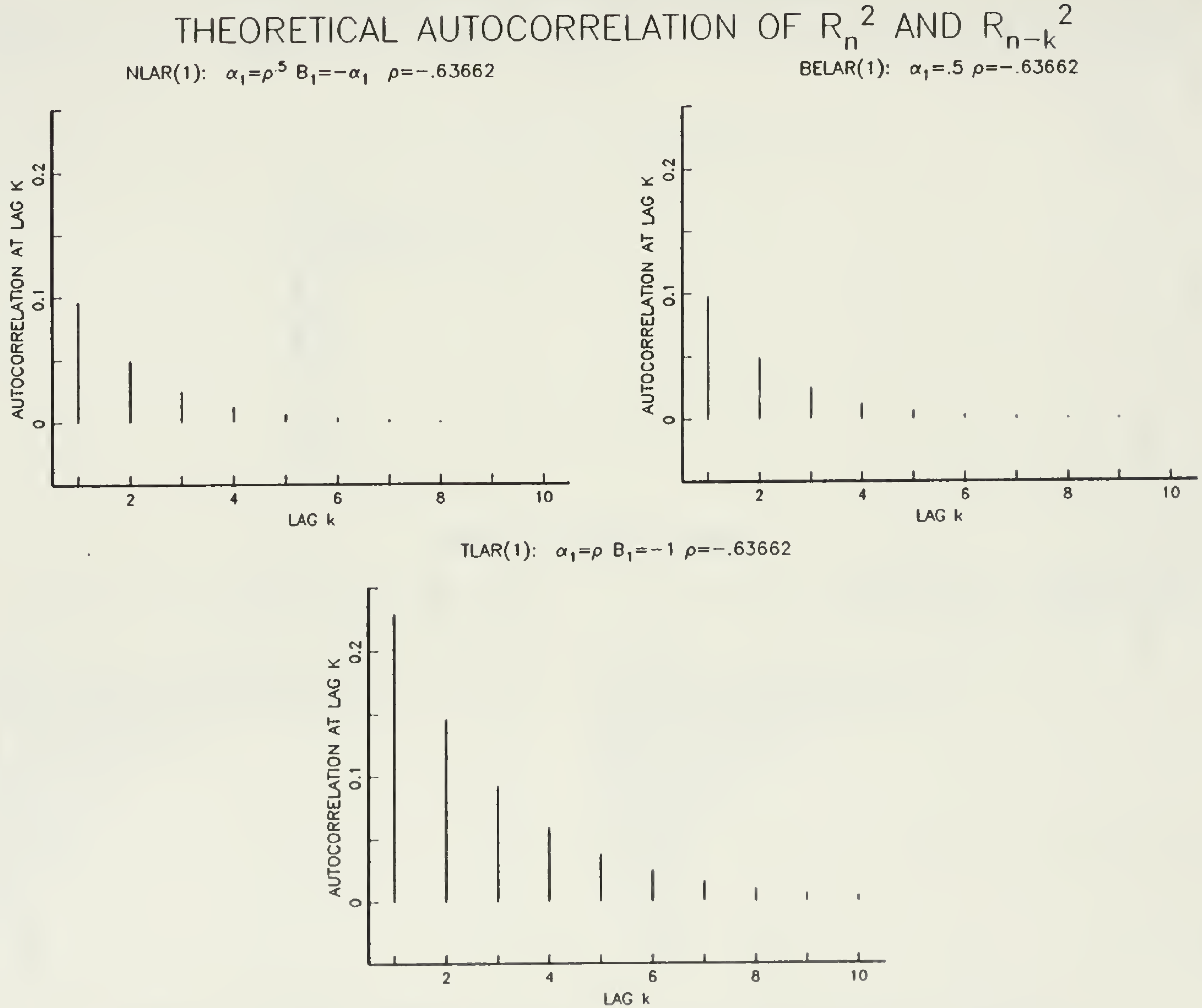


Figure IV.C.2. Theoretical Autocorrelation Functions of R_n^2 and R_{n-k}^2 for $k\geq 1$ for 3 RCA(1) Processes with $\rho(1)=-.63662$

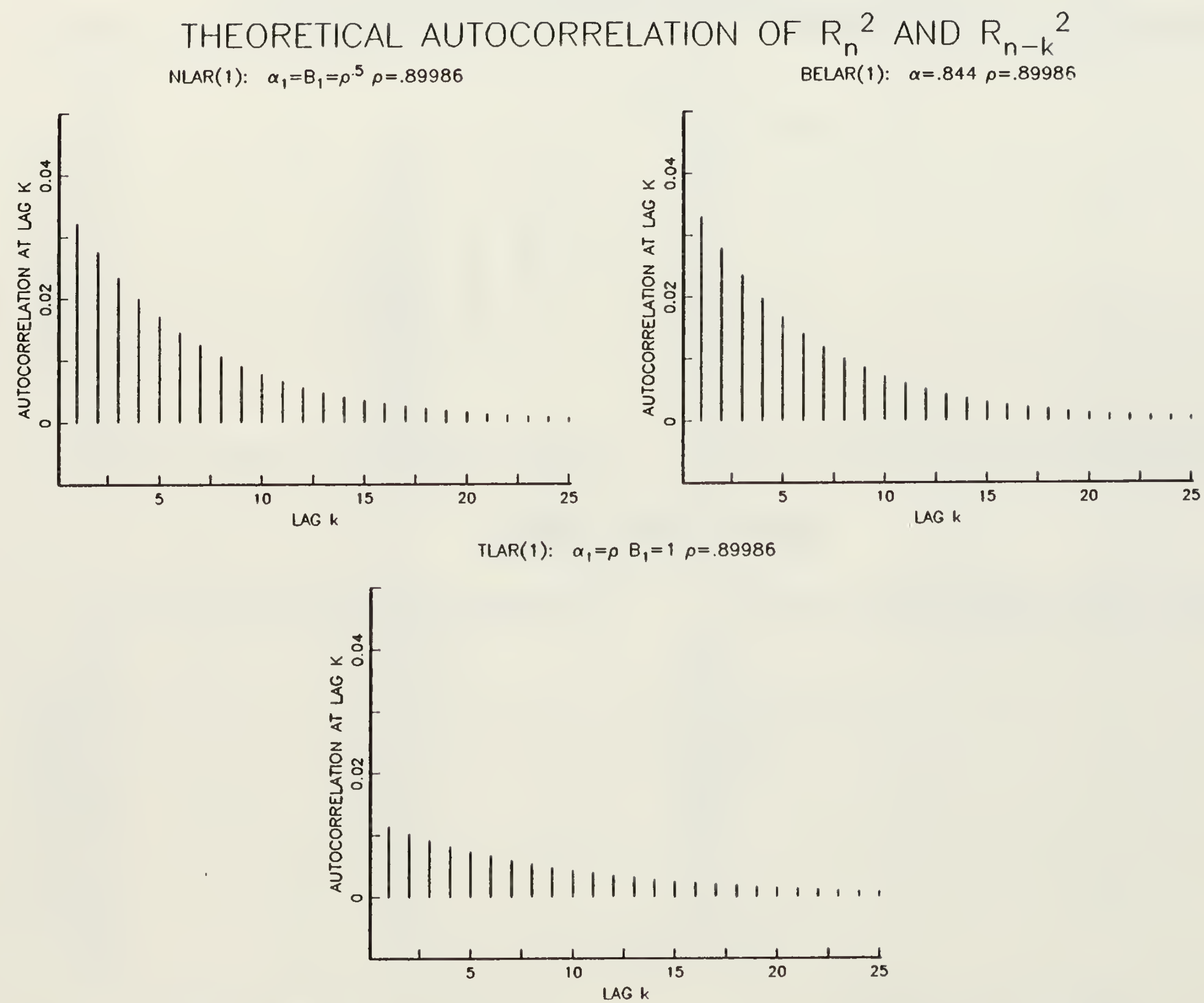


Figure IV.C.3. Theoretical Autocorrelation Functions of R_n^2 and R_{n-k}^2 for $k\geq1$ for 3 RCA(1) Processes with $\rho(1)=.89986$

IV.C.1 and IV.B.1. There is still little discrimination between the BELAR(1) model and the given non-boundary NLAR(1) model. However, the important point is that since the LAR(1) model is a linear AR(1) model, $\mathrm{Corr}(R_n^2, R_{n-k}^2) = 0$ for all values of ρ and for all $k = \pm 1, \pm 2, \ldots$.

V. EXTENSIONS AND OPEN PROBLEMS

A. INTRODUCTION

During the discussion in the previous chapters, possible extensions and/or unresolved issues have been mentioned. At this point, we conclude by summarizing some of the directions in which this research could be continued. There are still significant contributions to be made, particularly in parameter estimation, model development and applications.

B. ESTIMATION

There are several open questions and extensions in the area of parameter estimation and inference for this class of stochastic processes.

First, there is the need to obtain theoretical results substantiating the empirical results from the simulation of the maximum likelihood estimator (m.l.e.) of serial correlation in the TLAR(1) and the BELAR(1) processes. Several researchers have written on the subject of maximum likelihood estimation in dependent sequences. Much of this is assembled in the books by Basawa and Prakasa Rao [Ref. 42] and by Basawa and Scott [Ref. 43]. It is not known whether the conditions on the conditional densities are satisfied in the cases of these random coefficient AR(1) models to prove that the m.l.e. is consistent, asymptotically efficient or asymptotically Normal. Conditions for the existence of the m.l.e's

are generally extremely complicated and difficult to verify unless the log likelihood is absolutely continuous in the parameter space.

A second problem to resolve is that of existence and uniqueness of the maximum likelihood estimators of (α_1, β_1) in the NLAR(1) process. In this case, the log likelihood is definitely not differentiable with respect to the parameter, β_1, nor is it clear that there is a unique maximum. It appears from contour plots of the log-likelihood function over a grid of values in (α_1, β_1) coordinates that there is a unique local optimum within the region bounded by $0 < \alpha_1 < 1$ and $-1 < \beta_1 < 1$ for large enough samples of $\{X_n\}$. A non-linear optimization technique that uses only function values and not derivatives seems to be appropriate, since the log-likelihood function is not differentiable everywhere with respect to β_1.

A third problem involves the ℓ-Beta-Laplace AR(1) model. Except for the case when ℓ is assumed to be one (the BELAR(1) model), the likelihood function in (α, ℓ) has not been derived. This is primarily a numerical issue since neither the density of X_n for non-integer values of ℓ nor the conditional density of X_n given X_{n-1} for any values of $\ell > 0$ have closed-form expressions.

A fourth issue in estimation is to extend the maximum likelihood approach to include the joint estimation of the scale parameter of the marginal distribution to that of the shape parameter and the serial correlation coefficient. There is no reason to assume that the marginal distribution should always be a standard Laplace or standard ℓ-Laplace.

Finally, there is the issue of quantile estimation in the random coefficient models. Empirical results are given only for the BELAR(1) process for the distribution of the sample median. Theoretical results are related to mixing conditions. Based on a new mixing condition, which has been shown to be satisfied by linear AR(1) processes [Ref. 44], Gastwirth and Rubin derived the asymptotic Normal theory of quantile estimation for the linear LAR(1) process. The open question is whether the mixing condition of Gastwirth and Rubin is satisfied by any of the random coefficient models--NLAR(1), BELAR(1) or ℓ-Beta-Laplace AR(1).

C. MODEL DEVELOPMENT

Advances in modelling can be made in developing scalar models with p-th order autocorrelation structure, as well as bivariate autoregressive models.

An open question in the development of the NLARMA(p,q) family of models is the existence of the general class of models with p-th order autocorrelation structure--NLAR(p) for $p \geqq 3$; specifically, it is to derive the distribution of the i.i.d. innovation sequence $\{\varepsilon_n\}$. This is only known for the TLAR(p) subclass of a proposed NLAR(p) family.

A similar question is open for $p \geqq 2$ in the continuous random coefficient models with an ℓ-Laplace marginal distribution. The actual structure of the model, as well as the distribution of the innovation is in question.

There is also a need for multivariate time series in many fields of physical science. The NEAR(2) framework was used by Dewald and Lewis

[Ref. 24] to derive a bivariate exponential AR(1) model. Such an extension is also possible with the NLAR(2) model. Just how one estimates the eight possible parameters in such a model is an open question.

Related to the model development and parameter estimation is the need to identify particular models. Higher order residual analyses have been based on the linear residual $R_n = X_n - a_1 X_{n-1} - a_2 X_{n-2}$. Since the NLAR(2) model is only partially time reversible, it is possible that the reversed residual $\tilde{R}_n = X_n - a_1 X_{n+1} - a_2 X_{n+1}$ could be used in model identification as well. These were introduced by Lawrance and Lewis [Ref. 6, 45] but their use has not been explored in any context.

There is also the question of the effect that estimating a_1 and a_2 from $\{X_n\}$ will have on the sample autocorrelation of (R_n^2, R^2_{n-k}) and the cross-correlation of (X_n^3, R_{n-k}) in the fourth-order residual analyses proposed in Chapter IV.

D. APPLICATIONS

In Chapter I, several areas have been noted where the modelling is accomplished with heavy-tailed distributions, notably in voice and acoustics modelling, as well as in image coding. In these areas, the Laplace distribution and the symmetric Gamma distributions are widely used. There is the possibility that the ℓ-Laplace for $\ell < 1$ could also be a useful alternative to the symmetric Gamma. One advantage of the ℓ-Laplace distribution, which is the difference of two i.i.d. Gamma(ℓ, λ) is the simplicity of the form of the characteristic function.

Another field in which the ℓ-Laplace models could be useful is in the modelling of the directional components of wind speed. Models with skewed marginal distributions have been fitted to data and then transformed either to Normals (for example by Brown, Katz and Murphy [Ref. 46]), or to Exponentials by Lawrance and Lewis [Ref. 6]. In both of the cited papers, the data indicated that the wind was almost always blowing. The question is, however, how does one model wind velocity when there are long calm periods. This is a problem from Australia as related by T. Lewis in the discussion of the NEAR(2) model [Ref. 6]. As can be seen in Figure III.C.1, for small values of ℓ, highly correlated periods of calm and wind can be generated using the ℓ-Beta-Laplace AR(1) model.

The preceding examples demonstrate the opportunities for continued research and are not intended to narrow the focus of future endeavors.

VI. SUMMARY AND CONCLUSIONS

We have indicated by reference to the scientific literature that there are important application areas, especially in the physical sciences of time series whose marginal distributions are non-Normal. This feature, itself, presents new problems in the modelling, study and analysis.

For those areas where the non-Normality manifests itself primarily in the thickness (heaviness) of the tails of the marginal distributions, we have demonstrated that within the ℓ-Laplace family of distributions, there is an appropriate member with which to model phenomenon with a symmetric heavy-tailed marginal distribution. The ℓ-Laplace family has very thick tails when ℓ is small and a limiting Normal distribution as ℓ increases.

To account for serial dependence in the time series we have derived two families of random processes that extends the random coefficient approach to modelling non-Normal time series. The discrete random coefficient models (NLARMA(p,q)) have a Laplace marginal distribution and the continuous random coefficient models (ℓ-Beta-Laplace AR(1) and MA(q)) have an ℓ-Laplace marginal distribution. Both families are additive models and imitate the linear Gaussian models in that they exhibit the usual ARMA(p,q) correlation structure. The models are parametrically parsimonious, structurally simple and easy to generate on a computer.

We have also demonstrated that the fourth-order residual analyses based on the uncorrelated, but dependent sequence $\{R_n\}$ are appropriate and useful methods to discriminate between the discrete random coefficient and the

continuous random coefficient models when first, second and third-order properties are identical.

For the purposes of parameter estimation, we derived the joint probability density function. Numerical routines were written to maximize the likelihood function to estimate the serial correlation coefficient in the BELAR(1) and the TLAR(1) processes. Simulation results indicated that this estimator was more efficient and less biased than the least squares estimator derived from the linear residual.

Finally, we summarized some of the remaining issues in this field of non-Normal time series analysis. Extensions of the analyses in this thesis which need to be pursued are noted, along with possible applications in those previously mentioned fields of the physical sciences.

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